Army Research Laboratory

Perturbation Theory for the Landau-Lifshits-Gilbert Equation

by Frank Crowne

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1. Introduction

Although the literature on nonlinear magnetic effects is extensive, there is little quantitative information available on multi-frequency excitation of ferromagnetic systems, even at the level of perturbation theory. The calculations presented in this technical report constitute an attempt to remedy this situation. Because it is in the nature of remote sensing to involve small probe fields, the analysis needed is an ideal use of perturbation theory, cumbersome though it may be. It is hoped that their derivation will help to lay the groundwork for evaluating the usefulness of ferromagnetic resonance (FMR)-based detection of the nonlinear excitation of magnetic objects in a true remote-sensing environment, i.e., where large magnetic fields of the sort used in laboratories or medical equipment are unavailable.

2. The Landau-Lifshits and Landau-Lifshits-Gilbert Equations

In order to estimate the power radiated by FMR excitation of a finite magnetized body, it is necessary first to describe the dynamics of the magnetic dipole moment per unit volume $\vec{M}$ (i.e., the magnetization) within the body in the presence of an external time-dependent magnetic field. This dynamic problem is governed by the Landau-Lifshits equation (*1*):

$$\frac{\partial \vec{M}}{\partial t} = \gamma \vec{M} \times \vec{H}_{eff} - \frac{\alpha}{\left|\vec{M}_{st}\right|} \vec{M} \times \frac{\partial \vec{M}}{\partial t} \tag{1}$$

where $\vec{M}_{st}$ is the static magnetization of the material, γ is the gyromagnetic ratio, α is the damping constant, and $\vec{H}_{eff}$ is an effective magnetic field consisting of the DC anisotropy field $\vec{H}_a$ that fixes the direction of $\vec{M}_{st}$ in the material, the depolarization field due to the body's shape, and the external time-dependent applied field. Note that in the absence of an AC magnetic field there is no depolarization field, and so $\vec{M}_{st} \times \vec{H}_a = 0$ under DC conditions, i.e., there is no DC torque. The first term is the torque exerted by the effective magnetic field $\vec{H}_{eff}$, while the second term gives rise to damping via eddy currents. The form of this equation implies that $\vec{M} \bullet \frac{\partial \vec{M}}{\partial t} = 0 \Rightarrow \left|\vec{M}\right|^2$ is constant in time.

Let us solve this problem approximately using perturbation theory. It is more convenient to first put this equation in Landau-Lifshits-Gilbert (LLG) form (*2*): if we define the torque vector

$\vec{\Omega} = \gamma\vec{M} \times \vec{H}_{eff}$ and a dimensionless vector damping rate $\vec{Q} = \frac{\alpha}{\left|\vec{M}_{st}\right|}\vec{M}$, equation 1 can be rewritten as follows:

$$\frac{\partial\vec{M}}{\partial t} = \left(1+\alpha^2\right)^{-1}\left(\vec{\Omega} - \vec{Q}\times\vec{\Omega}\right) \tag{2}$$

Assume a coordinate system with the *z*-axis along the wire axis. We start by defining a small perturbing radio frequency (RF) magnetic field $\vec{h}$, which generates an RF magnetization $\vec{m}$ that is small compared to $\vec{M}_{st}$. The decay dynamics of this magnetization is strongly affected by the conservation of length of the total magnetization vector, whose endpoint is constrained to lie on a spherical surface at all times. Let us further specify the perturbed magnetization and effective magnetic field in the LLG equation as follows:

$$\vec{M} = \vec{M}_{st} + \vec{m} \tag{3}$$

and

$$\vec{H}_{eff} = \vec{H}_a + \vec{h} - N\vec{m}, \tag{4}$$

where $\vec{M}_{st} \times \vec{H}_a = 0$ and

$$N = \begin{pmatrix} \tilde{N} & 0 & 0 \\ 0 & \tilde{N} & 0 \\ 0 & 0 & N_z \end{pmatrix} \tag{5}$$

is the depolarization tensor. In this report, we consider a long wire parallel to the *z*-axis, for which $\tilde{N} = 1/2$ and $N_z \approx 0$. Following Antonenko et al. (*3*), we define a static susceptibility $\chi_{st} = \frac{\left|\vec{M}_{st}\right|}{\left|\vec{H}_a\right|}$. Then in rectangular coordinates with $\vec{M}_{st}$ and $\vec{H}_a$ along the positive z – axis, i.e., $\vec{M}_{st} = \left|\vec{M}_{st}\right|\hat{z}$ and $\vec{H}_a = H_a\hat{z}$, we obtain

$$\vec{H}_a = \chi_{st}^{-1}\left|\vec{M}_{st}\right|\hat{z} \Rightarrow \begin{cases} H_{eff,x} = h_x - \tilde{N}m_x \\ H_{eff,y} = h_y - \tilde{N}m_y \\ H_{eff,z} = h_z + \chi_{st}^{-1}\left|\vec{M}_{st}\right| \end{cases} \tag{6}$$

$$\Rightarrow \vec{\Omega} = \gamma\left(\vec{M}_{st} + \vec{\mathrm{m}}\right)\times \vec{H}_{eff} = \gamma\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \mathrm{m}_x & \mathrm{m}_y & \left|\vec{M}_{st}\right| + \mathrm{m}_z \\ \mathrm{h}_x - \tilde{N}\mathrm{m}_x & \mathrm{h}_y - \tilde{N}\mathrm{m}_y & \mathrm{h}_z + {\chi_{st}}^{-1}\left|\vec{M}_{st}\right| \end{vmatrix}$$
$$= \gamma\left\{\begin{array}{l} \hat{x}\left[-\left|\vec{M}_{st}\right|\mathrm{h}_y + \left({\chi_{st}}^{-1} + \tilde{N}\right)\left|\vec{M}_{st}\right|\mathrm{m}_y + \mathrm{m}_y\mathrm{h}_z - \mathrm{m}_z\mathrm{h}_y + \tilde{N}\mathrm{m}_z\mathrm{m}_y\right] \\ +\hat{y}\left[\left|\vec{M}_{st}\right|\mathrm{h}_x - \left({\chi_{st}}^{-1} + \tilde{N}\right)\left|\vec{M}_{st}\right|\mathrm{m}_x + \mathrm{m}_z\mathrm{h}_x - \mathrm{m}_x\mathrm{h}_z - \tilde{N}\mathrm{m}_z\mathrm{m}_x\right] \\ +\hat{z}\left[\mathrm{m}_x\mathrm{h}_y - \mathrm{m}_y\mathrm{h}_x\right] \end{array}\right\} \tag{7}$$

Introduce a scaled dimensionless magnetization $\vec{\mathrm{g}}$ such that $\vec{\mathrm{m}} = \left|\vec{M}_{st}\right|\vec{\mathrm{g}}$. Since $\vec{M}_{st} = \left|\vec{M}_{st}\right|\hat{z}$, the length constraint on $\vec{M}$ becomes

$$\begin{aligned} &\left|\vec{M}\right|^2 = \left|\vec{M}_{st} + \vec{\mathrm{m}}\right|^2 = \left|\vec{M}_{st}\right|^2 \Rightarrow \left|\hat{z} + \vec{\mathrm{g}}\right|^2 = \left(1 + \mathrm{g_z}\right)^2 + \mathrm{g_x}^2 + \mathrm{g_y}^2 = 1 \\ &\Rightarrow \mathrm{g_z} = -1 + \sqrt{1 - \mathrm{g_x}^2 - \mathrm{g_y}^2} \end{aligned} \tag{8}$$

Let $\vec{Q} = \alpha\left(\hat{z} + \vec{\mathrm{g}}\right)$. Then

$$\left|\vec{M}_{st}\right|\frac{\partial\vec{\mathrm{g}}}{\partial t} = \left(1 + \alpha^2\right)^{-1}\left(\vec{\Omega} - \vec{\mathrm{Q}}\times\vec{\Omega}\right) \tag{9}$$

and

$$\vec{\Omega} = \gamma\left|\vec{M}_{st}\right|\left\{\begin{array}{l} \hat{x}\left[-\mathrm{h}_y + \left({\chi_{st}}^{-1} + \tilde{N}\right)\left|\vec{M}_{st}\right|\mathrm{g}_y + \mathrm{g}_y\mathrm{h}_z - \mathrm{g}_z\mathrm{h}_y + \tilde{N}\left|\vec{M}_{st}\right|\mathrm{g}_z\mathrm{g}_y\right] \\ +\hat{y}\left[\mathrm{h}_x - \left({\chi_{st}}^{-1} + \tilde{N}\right)\left|\vec{M}_{st}\right|\mathrm{g}_x + \mathrm{g}_z\mathrm{h}_x - \mathrm{g}_x\mathrm{h}_z - \tilde{N}\left|\vec{M}_{st}\right|\mathrm{g}_z\mathrm{g}_x\right] \\ +\hat{z}\left[\mathrm{g}_x\mathrm{h}_y - \mathrm{g}_y\mathrm{h}_x\right] \end{array}\right\} \tag{10}$$

Defining the constants $\Lambda = \left({\chi_{st}}^{-1} + \tilde{N}\right)\left|\vec{M}_s\right|$, $\Theta = \left|\vec{M}_s\right|\tilde{N}$ lets us write the torque vector as

$$\vec{\Omega} = \gamma\left|\vec{M}_{st}\right|\left\{\begin{array}{l} \hat{x}\left[-\mathrm{h}_y + \Lambda\mathrm{g}_y + \mathrm{g}_y\mathrm{h}_z - \mathrm{g}_z\mathrm{h}_y + \Theta\mathrm{g}_z\mathrm{g}_y\right] \\ +\hat{y}\left[\mathrm{h}_x - \Lambda\mathrm{g}_x + \mathrm{g}_z\mathrm{h}_x - \mathrm{g}_x\mathrm{h}_z - \Theta\mathrm{g}_z\mathrm{g}_x\right] \\ +\hat{z}\left[\mathrm{g}_x\mathrm{h}_y - \mathrm{g}_y\mathrm{h}_x\right] \end{array}\right\} \tag{11}$$

The right side of the dynamic equation 2 can be treated as a “source” $\vec{S}$:

$$\vec{Q}\times\vec{\Omega}=\alpha\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ g_x & g_y & 1+g_z\\ \Omega_x & \Omega_y & \Omega_z\end{vmatrix}\Rightarrow\begin{cases} S_x=\left(\vec{\Omega}-\vec{Q}\times\vec{\Omega}\right)_x=\Omega_x-\alpha\left(g_y\Omega_z-[1+g_z]\Omega_y\right)\equiv\left|\vec{M}_{st}\right|\hat{S}_x\\ S_y=\left(\vec{\Omega}-\vec{Q}\times\vec{\Omega}\right)_y=\Omega_y-\alpha\left([1+g_z]\Omega_x-g_x\Omega_z\right)\equiv\left|\vec{M}_{st}\right|\hat{S}_y\\ S_z=\left(\vec{\Omega}-\vec{Q}\times\vec{\Omega}\right)_z=\Omega_z-\alpha\left(g_x\Omega_y-g_y\Omega_x\right)\equiv\left|\vec{M}_{st}\right|\hat{S}_z\end{cases}\tag{12}$$

where

$$\hat{S}_x=\gamma\left\{\begin{array}{l}-h_y+\Lambda g_y+g_yh_z-g_zh_y+\Theta g_zg_y\\ -\alpha g_y\left(g_xh_y-g_yh_x\right)\\ +\alpha[1+g_z]\left(h_x-\Lambda g_x+g_zh_x-g_xh_z-\Theta g_zg_x\right)\end{array}\right\}$$

$$\hat{S}_y=\gamma\left\{\begin{array}{l}h_x-\Lambda g_x+g_zh_x-g_xh_z-\Theta g_zg_x\\ -\alpha[1+g_z]\left(-h_y+\Lambda g_y+g_yh_z-g_zh_y+\Theta g_zg_y\right)\\ +\alpha g_x\left(g_xh_y-g_yh_x\right)\end{array}\right\}\tag{13}$$

$$\hat{S}_z=\gamma\left\{\begin{array}{l}g_xh_y-g_yh_x\\ -\alpha g_x\left(h_x-\Lambda g_x+g_zh_x-g_xh_z-\Theta g_zg_x\right)\\ +\alpha g_y\left(-h_y+\Lambda g_y+g_yh_z-g_zh_y+\Theta g_zg_y\right)\end{array}\right\}$$

Note that the vector $\hat{\vec{S}}$ has the dimensions of frequency.

It is advantageous to write the transverse system variables in vector form:

$$\vec{G}=\begin{pmatrix}g_x\\ g_y\end{pmatrix}\qquad\vec{H}=\begin{pmatrix}h_x\\ h_y\end{pmatrix}\tag{14}$$

Then, the equation for the transverse components of the source vector can be written as

$$\frac{\partial\vec{G}}{\partial t}=\left(1+\alpha^2\right)^{-1}\hat{\vec{S}}\tag{15}$$

where

$$\begin{aligned}\hat{\vec{S}}=\begin{pmatrix}\hat{S}_x\\ \hat{S}_y\end{pmatrix}&=\gamma\begin{pmatrix}\alpha[1+g_z]^2 & -[1+g_z]\\ [1+g_z] & \alpha[1+g_z]^2\end{pmatrix}\vec{H}-\gamma\begin{pmatrix}\alpha[1+g_z][\Lambda+h_z+\Theta g_z] & -[\Lambda+h_z+\Theta g_z-\alpha G\otimes H]\\ \Lambda+h_z+\Theta g_z-\alpha G\otimes H & \alpha[1+g_z][\Lambda+h_z+\Theta g_z]\end{pmatrix}\vec{G}\\ &=\gamma\begin{pmatrix}\alpha\left[1-G^2\right] & -\sqrt{1-G^2}\\ \sqrt{1-G^2} & \alpha\left[1-G^2\right]\end{pmatrix}\vec{H}-\gamma\begin{pmatrix}\alpha\sqrt{1-G^2}\left[\Lambda-\Theta+h_z+\Theta\sqrt{1-G^2}\right] & -\left[\Lambda-\Theta+h_z+\Theta\sqrt{1-G^2}-\alpha G\otimes H\right]\\ \Lambda-\Theta+h_z+\Theta\sqrt{1-G^2}-\alpha G\otimes H & \alpha\sqrt{1-G^2}\left[\Lambda-\Theta+h_z+\Theta\sqrt{1-G^2}\right]\end{pmatrix}\vec{G}\\ &\equiv\gamma\tilde{Z}\,\vec{H}-\gamma\tilde{R}\,\vec{G}\end{aligned}\tag{16}$$

Here $g_x{}^2 + g_y{}^2 \equiv G^2$ and $G \otimes H \equiv \hat{z} \cdot \vec{G} \times \vec{H} = g_x h_y - g_y h_x$. There is also an equation for the longitudinal part (z-component):

$$\left(1+\alpha^2\right)\frac{\partial}{\partial t} g_z + \gamma\alpha\left(\vec{G}\cdot\vec{H} - \Theta G^2\right) g_z = \gamma\left\{G \otimes H - \alpha \vec{G}\cdot\vec{H} + \alpha\left[\Lambda + h_z\right] G^2\right\} \tag{17}$$

where $\vec{G}\cdot\vec{H} \equiv g_x h_x + g_y h_y$, but the algebraic constraint on the length of the vector $\vec{g}$ makes it redundant.

Up to now, all the dynamic variables $g_x,\ g_y,\ g_z$ and $h_x,\ h_y,\ h_z$ have been real. In the spirit of the Holstein-Primakoff transformation (*4*), let us define the following (complex) "circular" basis for the transverse variables using the matrix

$$\tilde{C} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \Rightarrow \begin{cases} \vec{G} = \tilde{C}\vec{G} = \begin{pmatrix} g_x + i g_y \\ g_x - i g_y \end{pmatrix} = \begin{pmatrix} \mathcal{G} \\ \mathcal{G}^* \end{pmatrix} \\ \vec{H} = \tilde{C}\vec{H} = \begin{pmatrix} h_x + i h_y \\ h_x - i h_y \end{pmatrix} = \begin{pmatrix} H_c \\ H_c{}^* \end{pmatrix} \end{cases} \tag{18}$$

In this notation,

$$\begin{aligned} G \otimes H = g_x h_y - g_y h_x &= \frac{1}{4i}\left(\mathcal{G} + \mathcal{G}^*\right)\left(H_c - H_c{}^*\right) - \frac{1}{4i}\left(\mathcal{G} - \mathcal{G}^*\right)\left(H_c + H_c{}^*\right) \\ &= \frac{1}{2i}\left(\mathcal{G}^* H_c - \mathcal{G} H_c{}^*\right) = \mathrm{Im}\left(\mathcal{G}^* H_c\right) \end{aligned}. \tag{19}$$

The matrices $\tilde{Z}$ and $\tilde{R}$ in equation 3 are both of the form $\tilde{T}\left[A,B\right] = \begin{pmatrix} A\alpha & -B \\ B & A\alpha \end{pmatrix}$, which can be diagonalized by the similarity transformation defined by $\tilde{C}$:

$$\tilde{C}\tilde{T}\left[A,B\right]\tilde{C}^{-1} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} A\alpha & -B \\ B & A\alpha \end{pmatrix}\frac{1}{2i}\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \alpha A + iB & 0 \\ 0 & \alpha A - iB \end{pmatrix} \tag{20}$$

leading to the following expression for the source vector in the circular basis:

$$\begin{pmatrix} \hat{S}_+ \\ \hat{S}_- \end{pmatrix} = \begin{pmatrix} \hat{S}_x + i\hat{S}_y \\ \hat{S}_x - i\hat{S}_y \end{pmatrix} = \gamma\tilde{\Xi}\vec{H} - \gamma\tilde{\Pi}\vec{G} \tag{21}$$

where the matrices

$$\tilde{\Xi} = \begin{pmatrix} \sqrt{1-|\mathcal{G}|^2}\left[\alpha\sqrt{1-|\mathcal{G}|^2}+i\right] & 0 \\ 0 & \sqrt{1-|\mathcal{G}|^2}\left[\alpha\sqrt{1-|\mathcal{G}|^2}-i\right] \end{pmatrix} \tag{22}$$

and

$$\tilde{\Pi} = \begin{pmatrix} \begin{array}{l} \alpha\sqrt{1-|\mathcal{G}|^2}\left[\Lambda-\Theta+\mathrm{h}_z+\Theta\sqrt{1-|\mathcal{G}|^2}\right] \\ +i\left[\Lambda-\Theta+\mathrm{h}_z+\Theta\sqrt{1-|\mathcal{G}|^2}-\alpha\,\mathrm{Im}\left(\mathcal{G}^*H_c\right)\right] \end{array} & 0 \\ 0 & \begin{array}{l} \alpha\sqrt{1-|\mathcal{G}|^2}\left[\Lambda-\Theta+\mathrm{h}_z+\Theta\sqrt{1-|\mathcal{G}|^2}\right] \\ -i\left[\Lambda-\Theta+\mathrm{h}_z+\Theta\sqrt{1-|\mathcal{G}|^2}-\alpha\,\mathrm{Im}\left(\mathcal{G}^*H_c\right)\right] \end{array} \end{pmatrix} \tag{23}$$

are both diagonal. Since the nonzero elements of these matrices are complex conjugates of one another, the system reduces to a single complex-valued equation

$$\left(1+\alpha^2\right)\frac{\partial}{\partial t}\mathcal{G}+\gamma\begin{pmatrix} \alpha\sqrt{1-|\mathcal{G}|^2}\left[\Lambda-\Theta+\mathrm{h}_z+\Theta\sqrt{1-|\mathcal{G}|^2}\right] \\ +i\left[\Lambda-\Theta+\mathrm{h}_z+\Theta\sqrt{1-|\mathcal{G}|^2}-\alpha\,\mathrm{Im}\left(\mathcal{G}^*H_c\right)\right] \end{pmatrix}\mathcal{G} = \gamma\sqrt{1-|\mathcal{G}|^2}\left(\alpha\sqrt{1-|\mathcal{G}|^2}+i\right)H_c \tag{24}$$

where $\mathcal{G}$ and $\mathcal{G}^*$ are “magnon” variables (*5*). Note that the quantities $\sqrt{1-|\mathcal{G}|^2}$ and $\mathrm{Im}\left(\mathcal{G}^*H_c\right)$ are real.

A straightforward power-series expansion of this equation up to third order in $\mathcal{G}$ leads to the following perturbation equations:

$$\begin{aligned} &(\alpha-i)\frac{\partial}{\partial t}\mathcal{G}_1+\gamma\Lambda\mathcal{G}_1=\gamma H_c \\ &(\alpha-i)\frac{\partial}{\partial t}\mathcal{G}_2+\gamma\Lambda\mathcal{G}_2=-\gamma\mathrm{h}_z\mathcal{G}_1 \\ &(\alpha-i)\frac{\partial}{\partial t}\mathcal{G}_3+\gamma\Lambda\mathcal{G}_3=\frac{\gamma}{\alpha+i}\left\{\begin{array}{l} -(\alpha+i)\mathrm{h}_z\mathcal{G}_2-\frac{1}{2}(3\alpha+i)\left|\mathcal{G}_1\right|^2H_c+\alpha\frac{1}{2}\mathcal{G}_1{}^2H_c{}^* \\ +\frac{1}{2}\left(\alpha\Lambda+[\alpha+i]\Theta\right)\left|\mathcal{G}_1\right|^2\mathcal{G}_1 \end{array}\right\} \end{aligned} \tag{25}$$

The corresponding z-component equations are

$$
\begin{aligned}
&\mathrm{g}_z = -1+\sqrt{1-|\mathcal{G}|^2} = -1+\sqrt{1-\mathcal{G}^*\mathcal{G}} \approx -\frac{1}{2}\mathcal{G}_1{}^*\mathcal{G}_1 - \frac{1}{2}\left(\mathcal{G}_1{}^*\mathcal{G}_2 + \mathcal{G}_2{}^*\mathcal{G}_1\right) + \cdots \\
&\Rightarrow \begin{cases} \mathrm{g}_{z1} = 0 \\ \mathrm{g}_{z2} = -\dfrac{1}{2}\mathcal{G}_1{}^*\mathcal{G}_1 \\ \mathrm{g}_{z3} = -\dfrac{1}{2}\left(\mathcal{G}_1{}^*\mathcal{G}_2 + \mathcal{G}_2{}^*\mathcal{G}_1\right) \end{cases}
\end{aligned}
\tag{26}
$$

Note that according to these expressions an incident field with only h_z nonzero, i.e., polarized along the direction of static magnetization, cannot generate a *linear* response. This is in keeping with elementary considerations.

3. First-order Solution

To solve the equation

$$
(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_1 + \gamma\Lambda\mathcal{G}_1 = \gamma H_c \tag{27}
$$

we make the following decomposition: let

$$
\tilde{M} = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}
$$

Then

$$
\begin{aligned}
&\left(1+\alpha^2\right)\partial_t \begin{pmatrix} \mathrm{g}_x \\ \mathrm{g}_y \end{pmatrix} + \omega_B \tilde{M} \begin{pmatrix} \mathrm{g}_x \\ \mathrm{g}_y \end{pmatrix} = \gamma\tilde{M}\begin{pmatrix} S_x \\ S_y \end{pmatrix} \Rightarrow \left(1+\alpha^2\right)\partial_t \vec{\mathrm{g}} + \omega_B \tilde{M}\vec{\mathrm{g}} = \gamma\tilde{M}\vec{S} \\
&\tilde{C} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \Rightarrow \tilde{C}\begin{pmatrix} \mathrm{g}_x \\ \mathrm{g}_y \end{pmatrix} \square \begin{pmatrix} g_{c+} \\ g_{c-} \end{pmatrix} \Rightarrow \begin{cases} \begin{pmatrix} \mathrm{g}_x \\ \mathrm{g}_y \end{pmatrix} = \tilde{C}^{-1}\begin{pmatrix} g_{c+} \\ g_{c-} \end{pmatrix} = \dfrac{1}{2i}\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}\begin{pmatrix} g_{c+} \\ g_{c-} \end{pmatrix} \\ \begin{pmatrix} S_x \\ S_y \end{pmatrix} = \tilde{C}^{-1}\begin{pmatrix} S_{c+} \\ S_{c-} \end{pmatrix} = \dfrac{1}{2i}\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}\begin{pmatrix} S_{c+} \\ S_{c-} \end{pmatrix} \end{cases} \\
&\Rightarrow \left(1+\alpha^2\right)\partial_t \tilde{C}^{-1}\begin{pmatrix} g_{c+} \\ g_{c-} \end{pmatrix} + \omega_B \tilde{M}\tilde{C}^{-1}\begin{pmatrix} g_{c+} \\ g_{c-} \end{pmatrix} = \gamma\tilde{M}\tilde{C}^{-1}\begin{pmatrix} S_{c+} \\ S_{c-} \end{pmatrix} \\
&\Rightarrow \left(1+\alpha^2\right)\partial_t \begin{pmatrix} g_{c+} \\ g_{c-} \end{pmatrix} + \omega_B \tilde{C}\tilde{M}\tilde{C}^{-1}\begin{pmatrix} g_{c+} \\ g_{c-} \end{pmatrix} = \gamma\tilde{C}\tilde{M}\tilde{C}^{-1}\begin{pmatrix} S_{c+} \\ S_{c-} \end{pmatrix}
\end{aligned}
\tag{28}
$$

$$\tilde{C}\tilde{M}\tilde{C}^{-1} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix} \frac{1}{2i} \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \alpha + i & -1 + i\alpha \\ \alpha - i & -1 - i\alpha \end{pmatrix} \frac{1}{2i} \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2i} \begin{pmatrix} i\alpha - 1 - 1 + i\alpha & i\alpha - 1 + 1 - i\alpha \\ i\alpha + 1 - 1 - i\alpha & i\alpha + 1 + 1 + i\alpha \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} 2i\alpha - 2 & 0 \\ 0 & 2i\alpha + 2 \end{pmatrix} = \begin{pmatrix} \alpha + i & 0 \\ 0 & \alpha - i \end{pmatrix}$$

$$\Rightarrow \left(1 + \alpha^2\right) \partial_t \begin{pmatrix} g_{c+} \\ g_{c-} \end{pmatrix} + \omega_B \begin{pmatrix} [\alpha + i] g_{c+} \\ [\alpha - i] g_{c-} \end{pmatrix} = \gamma \begin{pmatrix} [\alpha + i] S_{c+} \\ [\alpha - i] S_{c-} \end{pmatrix} \tag{29}$$

$$\Rightarrow \begin{cases} \left(\partial_t + \dfrac{\omega_B}{\alpha - i} \right) g_{c+} = \dfrac{\gamma}{\alpha - i} S_{c+} \\ \left(\partial_t + \dfrac{\omega_B}{\alpha + i} \right) g_{c-} = \dfrac{\gamma}{\alpha + i} S_{c-} \end{cases}$$

The following is complex notation for non-Hermitian quantities:

$$\begin{pmatrix} S_x \\ S_y \end{pmatrix} = \begin{pmatrix} S_{xc} \\ S_{yc} \end{pmatrix} \cos \omega t + \begin{pmatrix} S_{xs} \\ S_{ys} \end{pmatrix} \sin \omega t = \begin{pmatrix} S_{xc} \\ S_{yc} \end{pmatrix} \frac{1}{2} \left[e^{i\omega t} + e^{-i\omega t} \right] + \begin{pmatrix} S_{xs} \\ S_{ys} \end{pmatrix} \frac{-i}{2} \left[e^{i\omega t} - e^{-i\omega t} \right]$$

$$= \frac{1}{2} \begin{pmatrix} S_{xc} + iS_{xs} \\ S_{yc} + iS_{ys} \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} S_{xc} - iS_{xs} \\ S_{yc} - iS_{ys} \end{pmatrix} e^{i\omega t} \equiv \begin{pmatrix} S_{x+} \\ S_{y+} \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} S_{x-} \\ S_{y-} \end{pmatrix} e^{i\omega t}$$

$$\Rightarrow \begin{pmatrix} S_{c+} \\ S_{c-} \end{pmatrix} \equiv \begin{pmatrix} S_{x+} + iS_{y+} \\ S_{x+} - iS_{y+} \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} S_{x-} + iS_{y-} \\ S_{x-} - iS_{y-} \end{pmatrix} e^{i\omega t} = \begin{pmatrix} S_{xc} + iS_{xs} + iS_{yc} - S_{ys} \\ S_{xc} + iS_{xs} - iS_{yc} + S_{ys} \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} S_{xc} - iS_{xs} + iS_{yc} + S_{ys} \\ S_{xc} - iS_{xs} - iS_{yc} - S_{ys} \end{pmatrix} e^{i\omega t}$$

$$= \begin{pmatrix} S_{xc} - S_{ys} + i\left[S_{xs} + S_{yc} \right] \\ S_{xc} + S_{ys} + i\left[S_{xs} - S_{yc} \right] \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} S_{xc} + S_{ys} - i\left[S_{xs} - S_{yc} \right] \\ S_{xc} - S_{ys} - i\left[S_{xs} + S_{yc} \right] \end{pmatrix} e^{i\omega t} \equiv \begin{pmatrix} S_+ \\ S_- \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} S_-{}^* \\ S_+{}^* \end{pmatrix} e^{i\omega t}$$

$$\Rightarrow \begin{cases} \left(\partial_t + \dfrac{\omega_B}{\alpha - i} \right) g_{c+} = \dfrac{\gamma}{\alpha - i} \left(S_+ e^{-i\omega t} + S_-{}^* e^{i\omega t} \right) \\ \left(\partial_t + \dfrac{\omega_B}{\alpha + i} \right) g_{c-} = \dfrac{\gamma}{\alpha + i} \left(S_- e^{-i\omega t} + S_+{}^* e^{i\omega t} \right) \end{cases} \tag{30}$$

The following is the transient solution:

$$p_\pm = \frac{\omega_B}{\alpha \mp i} \Rightarrow \begin{cases} g_{c+}(t) = A_+ e^{-p_+ t} + B_+ e^{-i\omega t} + C_+ e^{i\omega t} \\ g_{c-}(t) = A_- e^{-p_- t} + B_- e^{-i\omega t} + C_- e^{i\omega t} \end{cases}$$

$$\left(\partial_t + p_+\right) g_{c+} = \left(p_+ - i\omega\right) B_+ e^{-i\omega t} + \left(p_+ + i\omega\right) C_+ e^{i\omega t} = \frac{\gamma}{\alpha - i}\left(S_+ e^{-i\omega t} + S_-{}^* e^{i\omega t}\right)$$

$$\Rightarrow \begin{cases} B_+ = \dfrac{\gamma}{\alpha - i}\dfrac{S_+}{p_+ - i\omega} = \dfrac{\gamma S_+}{\omega_B - i\omega(\alpha - i)} = \dfrac{\gamma S_+}{\omega_B - \omega - i\alpha\omega} \\ C_+ = \dfrac{\gamma}{\alpha - i}\dfrac{S_-{}^*}{p_+ + i\omega} = \dfrac{\gamma S_-{}^*}{\omega_B + i\omega(\alpha - i)} = \dfrac{\gamma S_-{}^*}{\omega_B + \omega + i\alpha\omega} \end{cases}$$

$$\left(\partial_t + p_-\right) g_{c-} = \left(p_- - i\omega\right) B_- e^{-i\omega t} + \left(p_- + i\omega\right) C_- e^{i\omega t} = \frac{\gamma}{\alpha + i}\left(S_- e^{-i\omega t} + S_+{}^* e^{i\omega t}\right)$$

$$\Rightarrow \begin{cases} B_- = \dfrac{\gamma}{\alpha + i}\dfrac{S_-}{p_- - i\omega} = \dfrac{\gamma S_-}{\omega_B - i\omega(\alpha + i)} = \dfrac{\gamma S_-}{\omega_B + \omega - i\alpha\omega} \\ C_- = \dfrac{\gamma}{\alpha + i}\dfrac{S_+{}^*}{p_- + i\omega} = \dfrac{\gamma S_+{}^*}{\omega_B + i\omega(\alpha + i)} = \dfrac{\gamma S_+{}^*}{\omega_B - \omega + i\alpha\omega} \end{cases} \tag{31}$$

For the boundary condition at $t = 0$,

$$g_{c\pm}(0) \equiv g_{c\pm 0} = A_\pm + B_\pm + C_\pm \Rightarrow A_\pm = g_{c\pm 0} - B_\pm - C_\pm$$

$$\Rightarrow \begin{cases} g_{c+}(t) = g_{c+0} e^{-p_+ t} + \dfrac{\gamma S_+}{\omega_B - \omega - i\alpha\omega}\left(e^{-i\omega t} - e^{-p_+ t}\right) + \dfrac{\gamma S_-{}^*}{\omega_B + \omega + i\alpha\omega}\left(e^{i\omega t} - e^{-p_+ t}\right) \\ g_{c-}(t) = g_{c-0} e^{-p_- t} + \dfrac{\gamma S_-}{\omega_B + \omega - i\alpha\omega}\left(e^{-i\omega t} - e^{-p_- t}\right) + \dfrac{\gamma S_+{}^*}{\omega_B - \omega + i\alpha\omega}\left(e^{i\omega t} - e^{-p_- t}\right) \end{cases} \tag{32}$$

$$\begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left[g_{c+} + g_{c-}\right] \\ \frac{1}{2i}\left[g_{c+} - g_{c-}\right] \end{pmatrix}$$

$$p_- = p_+{}^* \Rightarrow g_{c-} = g_{c-0} e^{-p_- t} + \frac{\gamma}{\alpha+i}\frac{S_-}{p_- - i\omega}\left(e^{-i\omega t} - e^{-p_- t}\right) + \frac{\gamma}{\alpha+i}\frac{S_+{}^*}{p_- + i\omega}\left(e^{i\omega t} - e^{-p_- t}\right)$$

$$= g_{c-0} e^{-p_+{}^* t} + \left(\frac{\gamma}{\alpha-i}\right)^* \frac{S_-}{p_+{}^* - i\omega}\left(e^{-i\omega t} - e^{-p_+{}^* t}\right) + \left(\frac{\gamma}{\alpha-i}\right)^* \frac{S_+{}^*}{p_+{}^* + i\omega}\left(e^{i\omega t} - e^{-p_+{}^* t}\right)$$

$$= g_{c-0} e^{-p_+{}^* t} + \left(\frac{\gamma}{\alpha-i}\right)^* \frac{S_-}{p_+{}^* - i\omega}\left(e^{i\omega t} - e^{-p_+ t}\right)^* + \left(\frac{\gamma}{\alpha-i}\right)^* \frac{S_+{}^*}{p_+{}^* + i\omega}\left(e^{-i\omega t} - e^{-p_+ t}\right)^*$$

$$= g_{c-0}\left\{e^{-p_+ t}\right\}^* + \left(\frac{\gamma}{\alpha-i}\right)^* \left\{\frac{S_-{}^*}{p_+ + i\omega}\left(e^{i\omega t} - e^{-p_+ t}\right) + \frac{S_+}{p_+ - i\omega}\left(e^{-i\omega t} - e^{-p_+ t}\right)\right\}^* \tag{33}$$

$$g_{c-0} = g_{x0} - ig_{y0} = g_{c+0}{}^*$$

$$\Rightarrow g_{c-} = \left\{g_{c+0} e^{-p_+ t} + \frac{\gamma}{\alpha-i}\frac{S_-{}^*}{p_+ + i\omega}\left(e^{i\omega t} - e^{-p_+ t}\right) + \frac{\gamma}{\alpha-i}\frac{S_+}{p_+ - i\omega}\left(e^{-i\omega t} - e^{-p_+ t}\right)\right\}^* = g_{c+}{}^* \tag{34}$$

$$\Rightarrow \begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\left[\left(g_{x0} + ig_{y0}\right)e^{-p_+ t} + \frac{\gamma}{\alpha-i}\frac{S_-{}^*}{p_+ + i\omega}\left(e^{i\omega t} - e^{-p_+ t}\right) + \frac{\gamma}{\alpha-i}\frac{S_+}{p_+ - i\omega}\left(e^{-i\omega t} - e^{-p_+ t}\right)\right] \\ \mathrm{Im}\left[\left(g_{x0} + ig_{y0}\right)e^{-p_+ t} + \frac{\gamma}{\alpha-i}\frac{S_-{}^*}{p_+ + i\omega}\left(e^{i\omega t} - e^{-p_+ t}\right) + \frac{\gamma}{\alpha-i}\frac{S_+}{p_+ - i\omega}\left(e^{-i\omega t} - e^{-p_+ t}\right)\right] \end{pmatrix}$$

$$\text{Large t} \Rightarrow \begin{cases} g_{c+}(t) \to \dfrac{\gamma S_+}{\omega_B - \omega - i\alpha\omega} e^{-i\omega t} + \dfrac{\gamma S_-{}^*}{\omega_B + \omega + i\alpha\omega} e^{i\omega t} \\ g_{c-}(t) \to \dfrac{\gamma S_-}{\omega_B + \omega - i\alpha\omega} e^{-i\omega t} + \dfrac{\gamma S_+{}^*}{\omega_B - \omega + i\alpha\omega} e^{i\omega t} \end{cases} \tag{35}$$

$$\Rightarrow \begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\left[\dfrac{\gamma S_+}{\omega_B - \omega - i\alpha\omega} e^{-i\omega t} + \dfrac{\gamma S_-{}^*}{\omega_B + \omega + i\alpha\omega} e^{i\omega t}\right] \\ \mathrm{Im}\left[\dfrac{\gamma S_+}{\omega_B - \omega - i\alpha\omega} e^{-i\omega t} + \dfrac{\gamma S_-{}^*}{\omega_B + \omega + i\alpha\omega} e^{i\omega t}\right] \end{pmatrix}$$

The linear source terms are

$$
\begin{aligned}
&\begin{pmatrix} S_{\mathrm{x}} \\ S_{\mathrm{y}} \end{pmatrix} = \begin{pmatrix} \mathrm{h}_x \\ \mathrm{h}_y \end{pmatrix} = \begin{pmatrix} \mathrm{h}_{\mathrm{xc}} \\ \mathrm{h}_{\mathrm{yc}} \end{pmatrix} \cos\omega t + \begin{pmatrix} \mathrm{h}_{\mathrm{xs}} \\ \mathrm{h}_{\mathrm{ys}} \end{pmatrix} \sin\omega t = \frac{1}{2}\left\{ \begin{pmatrix} \mathrm{h}_{\mathrm{xc}} + i\mathrm{h}_{\mathrm{xs}} \\ \mathrm{h}_{\mathrm{yc}} + i\mathrm{h}_{\mathrm{ys}} \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} \mathrm{h}_{\mathrm{xc}} - i\mathrm{h}_{\mathrm{xs}} \\ \mathrm{h}_{\mathrm{yc}} - i\mathrm{h}_{\mathrm{ys}} \end{pmatrix} e^{i\omega t} \right\} \\
&\cong \begin{pmatrix} h_x \\ h_y \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} h_x{}^* \\ h_y{}^* \end{pmatrix} e^{i\omega t} \Rightarrow \begin{pmatrix} S_{c+} \\ S_{c-} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \left[\begin{pmatrix} h_x \\ h_y \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} h_x{}^* \\ h_y{}^* \end{pmatrix} e^{i\omega t} \right] \\
&= \begin{pmatrix} h_x + ih_y \\ h_x - ih_y \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} h_x{}^* + ih_y{}^* \\ h_x{}^* - ih_y{}^* \end{pmatrix} e^{i\omega t} \cong \begin{pmatrix} H_+ \\ H_- \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} H_-{}^* \\ H_+{}^* \end{pmatrix} e^{i\omega t} \\
&\mathcal{H}_x = \sqrt{\mathrm{h}_{\mathrm{xc}}{}^2 + \mathrm{h}_{\mathrm{xs}}{}^2}\,,\ \mathcal{H}_y = \sqrt{\mathrm{h}_{\mathrm{yc}}{}^2 + \mathrm{h}_{\mathrm{ys}}{}^2} \Rightarrow \begin{cases} h_x = \mathcal{H}_x e^{-i\theta_x} \\ h_y = \mathcal{H}_y e^{-i\theta_y} \end{cases}
\end{aligned} \tag{36}
$$

The two-frequency solution is

$$
\begin{aligned}
&\begin{pmatrix} \mathrm{g}_x + i\mathrm{g}_y \\ \mathrm{g}_x - i\mathrm{g}_y \end{pmatrix} = \begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_1{}^* \end{pmatrix} \\
&\Rightarrow \begin{cases} \left(\partial_t + \dfrac{\omega_B}{\alpha - i}\right)\mathcal{G}_{1\mathrm{s}} = \dfrac{\gamma}{\alpha - i}\left(H_+ e^{-i\omega_{\mathrm{s}} t} + H_-{}^* e^{i\omega_{\mathrm{s}} t}\right) \\ \left(\partial_t + \dfrac{\omega_B}{\alpha - i}\right)\mathcal{G}_{1\mathrm{p}} = \dfrac{\gamma}{\alpha - i}\left(H_+ e^{-i\omega_{\mathrm{p}} t} + H_-{}^* e^{i\omega_{\mathrm{p}} t}\right) \end{cases} \\
&\Rightarrow \begin{cases} \mathcal{G}_{1\mathrm{s}} = \dfrac{\gamma\left(h_{\mathrm{s}x} + ih_{\mathrm{s}y}\right)}{\omega_B - \omega_{\mathrm{s}} - i\alpha\omega_{\mathrm{s}}} e^{-i\omega_{\mathrm{s}} t} + \dfrac{\gamma\left(h_{\mathrm{s}x} - ih_{\mathrm{s}y}\right)^*}{\omega_B + \omega_{\mathrm{s}} + i\alpha\omega_{\mathrm{s}}} e^{i\omega_{\mathrm{s}} t} \\ \mathcal{G}_{1\mathrm{p}} = \dfrac{\gamma\left(h_{\mathrm{p}x} + ih_{\mathrm{p}y}\right)_+}{\omega_B - \omega_{\mathrm{p}} - i\alpha\omega_{\mathrm{p}}} e^{-i\omega_{\mathrm{p}} t} + \dfrac{\gamma\left(h_{\mathrm{p}x} - ih_{\mathrm{p}y}\right)^*}{\omega_B + \omega_{\mathrm{p}} + i\alpha\omega_{\mathrm{p}}} e^{i\omega_{\mathrm{p}} t} \end{cases}
\end{aligned} \tag{37}
$$

---

4. Second-order Solution

The solutions to

$$
(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_2 + \gamma\Lambda\mathcal{G}_2 = -\gamma \mathrm{h}_z \mathcal{G}_1 \tag{38}
$$

require the first-order solutions in the forms

$$
\begin{aligned}
\mathcal{G}_{1a,b} &= \Gamma_{a,b+} H_{a,b+} e^{-i\omega_{a,b} t} + \Gamma_{a,b-}{}^{*} H_{a,b-}{}^{*} e^{+i\omega_{a,b} t} \\
\mathcal{G}_{1a,b}{}^{*} &= \Gamma_{a,b-} H_{a,b-} e^{-i\omega_{a,b} t} + \Gamma_{a,b+}{}^{*} H_{a,b+}{}^{*} e^{+i\omega_{a,b} t}
\end{aligned} \tag{39}
$$

which gives the following expressions for the source terms:

$$
-\gamma h_z \mathcal{G}_1 = \left\{ \mathrm{Re}\left(h_{az} e^{-i\omega_a t} \right) + \mathrm{Re}\left(h_{bz} e^{-i\omega_b t} \right) \right\} \cdot \left\{ \mathcal{G}_{1a} + \mathcal{G}_{1b} \right\}
$$

$$
= -\frac{\gamma}{2} \left\{ h_{az} e^{-i\omega_a t} + h_{bz} e^{-i\omega_b t} + h_{az}{}^{*} e^{i\omega_a t} + h_{bz}{}^{*} e^{i\omega_b t} \right\} \left\{ \begin{array}{l} \Gamma_{a+} H_{a+} e^{-i\omega_a t} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} e^{+i\omega_a t} \\ + \Gamma_{b+} H_{b+} e^{-i\omega_b t} + \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_b t} \end{array} \right\}
$$

$$
= -\frac{\gamma}{2} \left[\begin{array}{l}
h_{az} \Gamma_{a+} H_{a+} e^{-2i\omega_a t} + h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az} \Gamma_{b+} H_{b+} e^{-i\omega_{sum} t} + h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{-i\omega_{dif} t} \\
+ h_{bz} \Gamma_{a+} H_{a+} e^{-i\omega_{sum} t} + h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} e^{+i\omega_{dif} t} + h_{bz} \Gamma_{b+} H_{b+} e^{-2i\omega_b t} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} \\
+ h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} e^{+2i\omega_a t} + h_{az}{}^{*} \Gamma_{b+} H_{b+} e^{+i\omega_{dif} t} + h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_{sum} t} \\
+ h_{bz}{}^{*} \Gamma_{a+} H_{a+} e^{-i\omega_{dif} t} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} e^{+i\omega_{sum} t} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} + h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+2i\omega_b t}
\end{array} \right] \tag{40}
$$

$$
= -\frac{\gamma}{2} \left[\begin{array}{l}
\left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-2i\omega_a t} + \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+2i\omega_a t} \\
+ h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \\
+ \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\omega_{sum} t} + \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\omega_{sum} t} \\
+ \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{-i\omega_{dif} t} + \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_{dif} t} \\
+ h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+2i\omega_b t} + h_{bz} \Gamma_{b+} H_{b+} e^{-2i\omega_b t}
\end{array} \right] \tag{41}
$$

where $\omega_{sum} = \omega_a + \omega_b$ and $\omega_{dif} = \omega_a - \omega_b$ are the sum and difference frequencies. The basic equation for the second-order perturbation $\mathcal{G}_2$ is

$$(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_2 + \omega_B \mathcal{G}_2 = -\frac{\gamma}{2}\left\{\begin{array}{l} \left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-2i\omega_a t} + \left(h_{az}{}^*\Gamma_{a-}{}^*H_{a-}{}^*\right)e^{+2i\omega_a t} \\ +h_{az}\Gamma_{a-}{}^*H_{a-}{}^* + h_{bz}\Gamma_{b-}{}^*H_{b-}{}^* + h_{az}{}^*\Gamma_{a+}H_{a+} + h_{bz}{}^*\Gamma_{b+}H_{b+} \\ +\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{sum}t} + \left(h_{az}{}^*\Gamma_{b-}{}^*H_{b-}{}^* + h_{bz}{}^*\Gamma_{a-}{}^*H_{a-}{}^*\right)e^{+i\omega_{sum}t} \\ +\left(h_{az}\Gamma_{b-}{}^*H_{b-}{}^* + h_{bz}{}^*\Gamma_{a+}H_{a+}\right)e^{-i\omega_{dif}t} + \left(h_{bz}\Gamma_{a-}{}^*H_{a-}{}^* + h_{az}{}^*\Gamma_{b+}H_{b+}\right)e^{+i\omega_{dif}t} \\ +h_{bz}{}^*\Gamma_{b-}{}^*H_{b-}{}^*e^{+2i\omega_b t} + h_{bz}\Gamma_{b+}H_{b+}e^{-2i\omega_b t} \end{array}\right\}$$

$$\Rightarrow \mathcal{G}_2 = -\left[\begin{array}{l} \Gamma_{2a+}\Gamma_{a+}h_{az}H_{a+}e^{-2i\omega_a t} + \Gamma_{2a-}\Gamma_{a-}{}^*h_{az}{}^*H_{a-}{}^*e^{+2i\omega_a t} \\ +\Gamma_0\left(h_{az}\Gamma_{a-}{}^*H_{a-}{}^* + h_{bz}\Gamma_{b-}{}^*H_{b-}{}^* + h_{az}{}^*\Gamma_{a+}H_{a+} + h_{bz}{}^*\Gamma_{b+}H_{b+}\right) \\ +\Gamma_{2b+}h_{bz}\Gamma_{b+}H_{b+}e^{-2i\omega_b t} + \Gamma_{2b-}h_{bz}{}^*\Gamma_{b-}{}^*H_{b-}{}^*e^{+2i\omega_b t} \\ +\Gamma_{sum+}\left(h_{az}\Gamma_{b-}{}^*H_{b-}{}^* + h_{bz}{}^*\Gamma_{a+}H_{a+}\right)e^{-i\omega_{sum}t} + \Gamma_{sum-}\left(h_{az}{}^*\Gamma_{b-}{}^*H_{b-}{}^* + h_{bz}{}^*\Gamma_{a-}{}^*H_{a-}{}^*\right)e^{+i\omega_{sum}t} \\ +\Gamma_{dif+}\left(h_{az}\Gamma_{b-}{}^*H_{b-}{}^* + h_{bz}{}^*\Gamma_{a+}H_{a+}\right)e^{-i\omega_{dif}t} + \Gamma_{dif-}\left(h_{bz}\Gamma_{a-}{}^*H_{a-}{}^* + h_{az}{}^*\Gamma_{b+}H_{b+}\right)e^{+i\omega_{dif}t} \end{array}\right] \quad (42)$$

where

$$\Gamma_0 = \frac{\gamma}{2}\frac{1}{\omega_B} \qquad \Gamma_{2a\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp 2\omega_a - i\alpha(2\omega_a)}$$

$$\Gamma_{2b\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp 2\omega_b - i\alpha(2\omega_b)} \qquad \Gamma_{s\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp \omega_{sum} - i\alpha(\omega_{sum})} \quad (43)$$

$$\Gamma_{d\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp \omega_{dif} - i\alpha(\omega_{dif})}$$

are new resonances associated with the various mixing products. For the z-component, we have

$$-\frac{1}{2}\left|\mathcal{G}_1\right|^2=-\frac{1}{2}\left|\mathcal{G}_{1a}+\mathcal{G}_{1b}\right|^2=-\frac{1}{2}\left\{\left|\mathcal{G}_{1a}\right|^2+\left|\mathcal{G}_{1b}\right|^2+2\operatorname{Re}\left(\mathcal{G}_{1a}{}^*\mathcal{G}_{1b}\right)\right\}$$

$$=-\frac{1}{2}\left\{\begin{array}{l}\left|\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^*H_{a-}{}^*e^{+i\omega_a t}\right|^2+\left|\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_b t}\right|^2\\ +2\operatorname{Re}\left(\left[\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^*H_{a-}{}^*e^{+i\omega_a t}\right]^*\left[\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_b t}\right]\right)\end{array}\right\}$$

$$=-\frac{1}{2}\left\{\begin{array}{l}\left|\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^*H_{a-}{}^*e^{+i\omega_a t}\right|^2+\left|\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_b t}\right|^2\\ +2\operatorname{Re}\left(\left[\Gamma_{a-}H_{a-}e^{-i\omega_a t}+\Gamma_{a+}{}^*H_{a+}{}^*e^{+i\omega_a t}\right]\left[\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_b t}\right]\right)\end{array}\right\}$$

$$=-\frac{1}{2}\left\{\begin{array}{l}\left|\Gamma_{a+}H_{a+}\right|^2+\left|\Gamma_{a-}{}^*H_{a-}{}^*\right|^2+2\operatorname{Re}\left(\Gamma_{a-}{}^*H_{a-}{}^*\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\right)\\ +\left|\Gamma_{b+}H_{b+}\right|^2+\left|\Gamma_{b-}{}^*H_{b-}{}^*\right|^2+2\operatorname{Re}\left(\Gamma_{b-}{}^*H_{b-}{}^*\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\right)\\ +2\operatorname{Re}\left(\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}e^{-i\omega_{sum}t}\right)+2\operatorname{Re}\left(\Gamma_{a-}H_{a-}\Gamma_{b-}{}^*H_{b-}{}^*e^{-i\omega_{dif}t}\right)\\ +2\operatorname{Re}\left(\Gamma_{a+}{}^*H_{a+}{}^*\Gamma_{b+}H_{b+}e^{+i\omega_{dif}t}\right)+2\operatorname{Re}\left(\Gamma_{a+}{}^*H_{a+}{}^*\Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_{sum}t}\right)\end{array}\right\}$$

$$\operatorname{Re}\left(\Gamma_{a+}{}^*H_{a+}{}^*\Gamma_{b-}{}^*H_{b-}{}^*e^{+i\omega_{sum}t}\right)=\operatorname{Re}\left(\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}e^{-i\omega_{sum}t}\right)$$

$$\operatorname{Re}\left(\Gamma_{a+}{}^*H_{a+}{}^*\Gamma_{b+}H_{b+}e^{+i\omega_{dif}t}\right)=\operatorname{Re}\left(\Gamma_{a+}H_{a+}\Gamma_{b+}{}^*H_{b+}{}^*e^{-i\omega_{dif}t}\right)$$

$$\Rightarrow g_{z2}=-\frac{1}{2}\left\{\begin{array}{l}\left|\Gamma_{a+}H_{a+}\right|^2+\left|\Gamma_{a-}{}^*H_{a-}{}^*\right|^2+2\operatorname{Re}\left(\Gamma_{a-}{}^*H_{a-}{}^*\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\right)\\ +\left|\Gamma_{b+}H_{b+}\right|^2+\left|\Gamma_{b-}{}^*H_{b-}{}^*\right|^2+2\operatorname{Re}\left(\Gamma_{b-}{}^*H_{b-}{}^*\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\right)\\ +2\operatorname{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_{sum}t}\right)\\ +2\operatorname{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^*H_{b-}{}^*+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^*H_{b+}{}^*\right]e^{-i\omega_{dif}t}\right)\end{array}\right\} \tag{44}$$

Two frequencies of special interest to remote sensing are the sum and difference frequencies. The magnetization amplitudes at these frequencies are given by

$$\mathcal{G}_{2sum} = -\left[\Gamma_{sum+}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{sum}t} + \Gamma_{sum-}\left(h_{az}\Gamma_{b-}H_{b-} + h_{bz}\Gamma_{a-}H_{a-}\right)^{*}e^{+i\omega_{sum}t}\right]$$
$$g_{2sum,x} = \mathrm{Re}\,\mathcal{G}_{2sum} \qquad (45)$$
$$g_{2sum,y} = \mathrm{Im}\,\mathcal{G}_{2sum}$$
$$g_{2sum,z} = -\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_{sum}t}\right)$$

and

$$\mathcal{G}_{2dif} = -\left[\Gamma_{dif+}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{dif}t} + \Gamma_{dif-}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_{dif}t}\right]$$
$$g_{2dif,x} = \mathrm{Re}\,\mathcal{G}_{2dif} \qquad (46)$$
$$g_{2dif,y} = \mathrm{Im}\,\mathcal{G}_{2dif}$$
$$g_{2dif,z} = -\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_{dif}t}\right)$$

5. Third-order Solution

The time-domain equation of the third-order system has a complicated source term:

$$(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_3 + \omega_B\mathcal{G}_3 = \frac{\gamma}{\alpha + i}\begin{Bmatrix} -\frac{1}{2}(3\alpha + i)\left|\mathcal{G}_1\right|^2\mathcal{H} + \alpha\frac{1}{2}\mathcal{G}_1{}^2\mathcal{H}^* \\ +\frac{1}{2}\left(\alpha\Lambda + [\alpha + i]\Theta\right)\left|\mathcal{G}_1\right|^2\mathcal{G}_1 - (\alpha + i)h_z\mathcal{G}_2 \end{Bmatrix} \qquad (47)$$
$$\equiv \frac{\gamma}{2}\left\{-\left(1 + \frac{2\alpha}{\alpha + i}\right)\mathbb{S}_1 + \frac{\alpha}{\alpha + i}\mathbb{S}_2 + \left(\frac{\alpha}{\alpha + i}\Lambda + \Theta\right)\mathbb{S}_3 - 2\mathbb{S}_4\right\}$$

To evaluate $\mathcal{G}_3$, we must first express the various products in multi-frequency form. This leads to the following linear inhomogeneous equation:

$$
(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_3 + \omega_B \mathcal{G}_3 = \frac{\gamma}{2}\begin{pmatrix}
S_{3,a+}e^{-i\omega_a t} + S_{3,a-}e^{+i\omega_a t} + S_{3,b+}e^{-i\omega_b t} + S_{3,b-}e^{+i\omega_b t} \\
+S_{3,3a+}e^{-3i\omega_a t} + S_{3,3a-}e^{+3i\omega_a t} + S_{3,3b+}e^{-3i\omega_b t} + S_{3,3b-}e^{+3i\omega_b t} \\
+S_{3,2b+a,+}e^{-i\left(2\omega_b + \omega_a\right)t} + S_{3,2b+a,-}e^{+i\left(2\omega_b + \omega_a\right)t} \\
+S_{3,2a+b,+}e^{-i\left(2\omega_a + \omega_b\right)t} + S_{3,2a+b,-}e^{+i\left(2\omega_a + \omega_b\right)t} \\
+S_{3,2b-a,+}e^{-i\left(2\omega_b - \omega_a\right)t} + S_{3,2b-a,-}e^{+i\left(2\omega_b - \omega_a\right)t} \\
+S_{3,2a-b,+}e^{-i\left(2\omega_a - \omega_b\right)t} + S_{3,2a-b,-}e^{+i\left(2\omega_a - \omega_b\right)t}
\end{pmatrix} \quad (48)
$$

which is easily solved:

$$
\mathcal{G}_3 = \begin{pmatrix}
\Gamma_{a+}S_{3,a+}e^{-i\omega_a t} + \Gamma_{a-}S_{3,a-}e^{+i\omega_a t} + \Gamma_{b+}S_{3,b+}e^{-i\omega_b t} + \Gamma_{b-}S_{3,b-}e^{+i\omega_b t} \\
+\Gamma_{3a+}S_{3,3a+}e^{-3i\omega_a t} + \Gamma_{3a-}S_{3,3a-}e^{+3i\omega_a t} + \Gamma_{3b+}S_{3,3b+}e^{-3i\omega_b t} + \Gamma_{3b-}S_{3,3b-}e^{+3i\omega_b t} \\
+\Gamma_{2b+a+}S_{3,2b+a,+}e^{-i\left(2\omega_b + \omega_a\right)t} + \Gamma_{2b+a,-}S_{3,2b+a,-}e^{+i\left(2\omega_b + \omega_a\right)t} \\
+\Gamma_{2a+b+}S_{3,2a+b,+}e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{2a+b,-}S_{3,2a+b,-}e^{+i\left(2\omega_a + \omega_b\right)t} \\
+\Gamma_{2b-a+}S_{3,2b-a,+}e^{-i\left(2\omega_b - \omega_a\right)t} + \Gamma_{2b-a,-}S_{3,2b-a,-}e^{+i\left(2\omega_b - \omega_a\right)t} \\
+\Gamma_{2amb+}S_{3,2a-b,+}e^{-i\left(2\omega_a - \omega_b\right)t} + \Gamma_{2a-b,-}S_{3,2a-b,-}e^{+i\left(2\omega_a - \omega_b\right)t}
\end{pmatrix} \quad (49)
$$

where

$$
\begin{aligned}
&\Gamma_{a\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp \omega_a - i\alpha\omega_a} && \Gamma_{b\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp \omega_b - i\alpha\omega_b}\\
&\Gamma_{3a\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp 3\omega_a - i\alpha(3\omega_a)} && \Gamma_{3b\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp 3\omega_b - i\alpha(3\omega_b)}\\
&\Gamma_{2b+a,\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp (2\omega_b + \omega_a) - i\alpha(2\omega_b + \omega_a)} && \Gamma_{2b-a,\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp (2\omega_b - \omega_a) - i\alpha(2\omega_b - \omega_a)}\\
&\Gamma_{2a+b,\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp (2\omega_a + \omega_b) - i\alpha(2\omega_a + \omega_b)} && \Gamma_{2a-b,\pm} = \frac{\gamma}{2}\frac{1}{\omega_B \mp (2\omega_a - \omega_b) - i\alpha(2\omega_a - \omega_b)}
\end{aligned}
\tag{50}
$$

Note that the third-order source terms give rise to three distinct effects: frequency tripling, amplitude-dependence of the response functions at the drive frequencies, and intermodulation products.

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6. Grouping Third-order Magnetization Terms

The only terms of interest here are the intermod terms, which are color-coded to make the algebra easier to follow.

$$
\begin{aligned}
\mathbb{S}_1 &= \left|\mathcal{G}_1\right|^2 \mathcal{H}\\
&= \left\{
\begin{aligned}
&\left|\Gamma_{a+}H_{a+}\right|^2 + \left|\Gamma_{a-}{}^*H_{a-}{}^*\right|^2 + 2\,\mathrm{Re}\left(\Gamma_{a-}{}^*H_{a-}{}^*\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\right)\\
&+\left|\Gamma_{b+}H_{b+}\right|^2 + \left|\Gamma_{b-}{}^*H_{b-}{}^*\right|^2 + 2\,\mathrm{Re}\left(\Gamma_{b-}{}^*H_{b-}{}^*\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\right)\\
&+2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_s t}\right)\\
&+2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^*H_{b-}{}^* + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^*H_{b+}{}^*\right]e^{-i\omega_d t}\right)
\end{aligned}
\right\}
\frac{1}{2}\left\{
\begin{aligned}
&H_{a+}e^{-i\omega_a t} + H_{a-}{}^*e^{+i\omega_a t}\\
&+H_{b+}e^{-i\omega_b t} + H_{b-}{}^*e^{+i\omega_b t}
\end{aligned}
\right\}\\
&\equiv \frac{1}{2}\sum_{m=1}^{5} \mathrm{X}_m
\end{aligned}
\tag{51}
$$

$$
\mathrm{X}_1 = \left\{\left|\Gamma_{a+}H_{a+}\right|^2 + \left|\Gamma_{a-}{}^*H_{a-}{}^*\right|^2 + \left|\Gamma_{b+}H_{b+}\right|^2 + \left|\Gamma_{b-}{}^*H_{b-}{}^*\right|^2\right\}\left\{H_{a+}e^{-i\omega_a t} + H_{a-}{}^*e^{+i\omega_a t} + H_{b+}e^{-i\omega_b t} + H_{b-}{}^*e^{+i\omega_b t}\right\}
\tag{52}
$$

$$X_2 = 2\,\mathrm{Re}\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\right)\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$=\left\{\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\right\}\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$=\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}H_{a-}{}^{*}e^{+i\omega_a t}$$

$$+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}H_{b+}e^{-i\omega_b t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}H_{b-}{}^{*}e^{+i\omega_b t}$$

$$+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}H_{a-}{}^{*}e^{+i\omega_a t}$$

$$+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}H_{b+}e^{-i\omega_b t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}H_{b-}{}^{*}e^{+i\omega_b t}$$

$$=\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{a+}e^{-3i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{a-}{}^{*}e^{-i\omega_a t}$$

$$+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{b+}e^{-i\left(2\omega_a+\omega_b\right)t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{b-}{}^{*}e^{-i\left(2\omega_a-\omega_b\right)t}$$

$$+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{a+}e^{+i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{a-}{}^{*}e^{+3i\omega_a t} \tag{53}$$

$$+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{b+}e^{+i\left(2\omega_a-\omega_b\right)t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t}$$

$$X_3 = 2\,\mathrm{Re}\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\right)\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$=\left\{\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\right\}\left\{H_{a+}e^{-i\omega_a t}+H_{a-}{}^{*}e^{+i\omega_a t}+H_{b+}e^{-i\omega_b t}+H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$=\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}H_{a+}e^{-i\omega_a t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t}$$

$$+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t}$$

$$+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}H_{a+}e^{-i\omega_a t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t}$$

$$+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t}$$

$$=\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{a+}e^{-i\left(2\omega_b+\omega_a\right)t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{a-}{}^{*}e^{-i\left(2\omega_b-\omega_a\right)t}$$

$$+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{b+}e^{-3i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{b-}{}^{*}e^{-i\omega_b t}$$

$$+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{a+}e^{+i\left(2\omega_b-\omega_a\right)t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t} \tag{54}$$

$$+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{b+}e^{+i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{b-}{}^{*}e^{+3i\omega_b t}$$

$$X_4 = 2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_s t}\right)\left\{H_{a+}e^{-i\omega_a t} + H_{a-}{}^{*}e^{+i\omega_a t} + H_{b+}e^{-i\omega_b t} + H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$= \left\{\begin{array}{l}\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t} \\ +\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\end{array}\right\}\left\{H_{a+}e^{-i\omega_a t} + H_{a-}{}^{*}e^{+i\omega_a t} + H_{b+}e^{-i\omega_b t} + H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$\begin{aligned}
&= \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}H_{a+}e^{-i\omega_a t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}H_{b+}e^{-i\omega_b t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}H_{a+}e^{-i\omega_a t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}H_{b+}e^{-i\omega_b t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t}
\end{aligned} \tag{55}$$

$$\begin{aligned}
&= \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{a+}e^{-i\left(2\omega_a + \omega_b\right)t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{a-}{}^{*}e^{-i\omega_b t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{b+}e^{-i\left(2\omega_b + \omega_a\right)t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{b-}{}^{*}e^{-i\omega_a t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{a+}e^{+i\omega_b t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{a-}{}^{*}e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{b+}e^{+i\omega_a t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{b-}{}^{*}e^{+i\left(2\omega_b + \omega_a\right)t}
\end{aligned} \tag{56}$$

$$X_5 = 2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_d t}\right)\left\{H_{a+}e^{-i\omega_a t} + H_{a-}{}^{*}e^{+i\omega_a t} + H_{b+}e^{-i\omega_b t} + H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$= \left\{\begin{array}{l}\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t} \\ +\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\end{array}\right\}\left\{H_{a+}e^{-i\omega_a t} + H_{a-}{}^{*}e^{+i\omega_a t} + H_{b+}e^{-i\omega_b t} + H_{b-}{}^{*}e^{+i\omega_b t}\right\}$$

$$\begin{aligned}
&= \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}H_{a+}e^{-i\omega_a t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}H_{b+}e^{-i\omega_b t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}H_{a+}e^{-i\omega_a t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}H_{a-}{}^{*}e^{+i\omega_a t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}H_{b+}e^{-i\omega_b t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}H_{b-}{}^{*}e^{+i\omega_b t}
\end{aligned} \tag{57}$$

$$\begin{aligned}
&= \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{a+}e^{-i\left(2\omega_a - \omega_b\right)t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{a-}{}^{*}e^{+i\omega_b t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{b+}e^{-i\omega_a t} \\
&+ \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{b-}{}^{*}e^{+i\left(2\omega_b - \omega_a\right)t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{a+}e^{-i\omega_b t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{a-}{}^{*}e^{+i\left(2\omega_a - \omega_b\right)t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{b+}e^{-i\left(2\omega_b - \omega_a\right)t} \\
&+ \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{b-}{}^{*}e^{+i\omega_a t}
\end{aligned} \tag{58}$$

$$\mathbb{S}_2 = \mathcal{G}_1{}^2 \mathcal{H}^* = \begin{Bmatrix} \Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^* H_{a-}{}^* e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^* H_{b-}{}^* e^{+i\omega_b t} \end{Bmatrix}^2 \frac{1}{2} \begin{Bmatrix} H_{a+}e^{-i\omega_a t} + H_{a-}{}^* e^{+i\omega_a t} \\ +H_{b+}e^{-i\omega_b t} + H_{b-}{}^* e^{+i\omega_b t} \end{Bmatrix}^*$$

$$= \frac{1}{2} \begin{Bmatrix} \Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^* H_{a-}{}^* e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^* H_{b-}{}^* e^{+i\omega_b t} \end{Bmatrix}^2 \begin{Bmatrix} H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} \\ +H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t} \end{Bmatrix}$$

$$= \frac{1}{2} \begin{Bmatrix} \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} + 2\Gamma_{a+}H_{a+}\Gamma_{a-}{}^* H_{a-}{}^* + \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} e^{+2i\omega_a t} \\ +2\left(\Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^* H_{a-}{}^* e^{+i\omega_a t}\right)\left(\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^* H_{b-}{}^* e^{+i\omega_b t}\right) \\ +\Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} + 2\Gamma_{b-}{}^* H_{b-}{}^* \Gamma_{b+}H_{b+} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} \end{Bmatrix} \begin{Bmatrix} H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} \\ +H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t} \end{Bmatrix}$$

$$= \frac{1}{2} \begin{Bmatrix} \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} + 2\Gamma_{a+}H_{a+}\Gamma_{a-}{}^* H_{a-}{}^* + \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} e^{+2i\omega_a t} \\ +2\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}e^{-i\omega_s t} + 2\Gamma_{a+}H_{a+}\Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_d t} \\ +2\Gamma_{a-}{}^* H_{a-}{}^* \Gamma_{b+}H_{b+}e^{+i\omega_d t} + 2\Gamma_{a-}{}^* H_{a-}{}^* \Gamma_{b-}{}^* H_{b-}{}^* e^{+i\omega_s t} \\ +\Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} + 2\Gamma_{b-}{}^* H_{b-}{}^* \Gamma_{b+}H_{b+} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} \end{Bmatrix} \begin{Bmatrix} H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} \\ +H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t} \end{Bmatrix} \quad (59)$$

$$= \frac{1}{2} \begin{Bmatrix} \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} + 2\Gamma_{a+}H_{a+}\Gamma_{a-}{}^* H_{a-}{}^* + \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} e^{+2i\omega_a t} \\ +2\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}e^{-i\omega_s t} + 2\Gamma_{a+}H_{a+}\Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_d t} \\ +2\Gamma_{a-}{}^* H_{a-}{}^* \Gamma_{b+}H_{b+}e^{+i\omega_d t} + 2\Gamma_{a-}{}^* H_{a-}{}^* \Gamma_{b-}{}^* H_{b-}{}^* e^{+i\omega_s t} \\ +\Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} + 2\Gamma_{b-}{}^* H_{b-}{}^* \Gamma_{b+}H_{b+} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} \end{Bmatrix} \begin{Bmatrix} H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} \\ +H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t} \end{Bmatrix} \quad (60)$$

$$
=\frac{1}{2}\left\{\begin{aligned}
&\Gamma_{a+}{}^{2}H_{a+}{}^{2}e^{-2i\omega_a t}\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]\\
&+\left(2\Gamma_{a+}H_{a+}\Gamma_{a-}{}^{*}H_{a-}{}^{*}+2\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\right)\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]\\
&+\Gamma_{a-}{}^{*2}H_{a-}{}^{*2}e^{+2i\omega_a t}\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]\\
&+2\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}e^{-i\omega_s t}\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]\\
&+2\Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\omega_d t}\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]\\
&+2\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}e^{+i\omega_d t}\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]\\
&+2\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_s t}\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]\\
&+\Gamma_{b+}{}^{2}H_{b+}{}^{2}e^{-2i\omega_b t}\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]\\
&+\Gamma_{b-}{}^{*2}H_{b-}{}^{*2}e^{+2i\omega_b t}\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]
\end{aligned}\right\} \quad (61)
$$

$$
=\frac{1}{2}\sum_{m=1}^{9}\Upsilon_m
$$

$$
\Upsilon_1=\left(2\Gamma_{a+}H_{a+}\Gamma_{a-}{}^{*}H_{a-}{}^{*}+2\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\right)\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right] \quad (62)
$$

$$\Upsilon_2 = \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$
$$= \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} H_{a-}e^{-i\omega_a t} + \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} H_{a+}{}^* e^{+i\omega_a t}$$
$$+\Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} H_{b-} e^{-i\omega_b t} + \Gamma_{a+}{}^2 H_{a+}{}^2 e^{-2i\omega_a t} H_{b+}{}^* e^{+i\omega_b t}$$
$$= \Gamma_{a+}{}^2 H_{a+}{}^2 H_{a-} e^{-3i\omega_a t} + \Gamma_{a+}{}^2 H_{a+}{}^2 H_{a+}{}^* e^{-i\omega_a t}$$
$$+\Gamma_{a+}{}^2 H_{a+}{}^2 H_{b-} e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{a+}{}^2 H_{a+}{}^2 H_{b+}{}^* e^{-i\left(2\omega_a - \omega_b\right)t}$$
$$\Upsilon_3 = \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} e^{+2i\omega_a t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$
$$= \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} H_{a-} e^{+i\omega_a t} + \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} H_{a+}{}^* e^{+3i\omega_a t}$$
$$+\Gamma_{a-}{}^{*2} H_{a-}{}^{*2} H_{b-} e^{+i\left(2\omega_a - \omega_b\right)t} + \Gamma_{a-}{}^{*2} H_{a-}{}^{*2} H_{b+}{}^* e^{+i\left(2\omega_a + \omega_b\right)t} \tag{63}$$
$$\Upsilon_4 = \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_s t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$
$$= \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_a t} e^{-i\omega_b t} H_{a-} e^{-i\omega_a t} + \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_a t} e^{-i\omega_b t} H_{a+}{}^* e^{+i\omega_a t}$$
$$+\Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_a t} e^{-i\omega_b t} H_{b-} e^{-i\omega_b t} + \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} e^{-i\omega_a t} e^{-i\omega_b t} H_{b+}{}^* e^{+i\omega_b t}$$
$$= \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} H_{a-} e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} H_{a+}{}^* e^{-i\omega_b t}$$
$$+\Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} H_{b-} e^{-i\left(2\omega_b + \omega_a\right)t} + \Gamma_{a+} H_{a+} \Gamma_{b+} H_{b+} H_{b+}{}^* e^{-i\omega_a t}$$
$$\Upsilon_5 = \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_d t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$
$$= \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_a t} e^{+i\omega_b t} H_{a-} e^{-i\omega_a t} + \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_a t} e^{+i\omega_b t} H_{a+}{}^* e^{+i\omega_a t}$$
$$+\Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_a t} e^{+i\omega_b t} H_{b-} e^{-i\omega_b t} + \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* e^{-i\omega_a t} e^{+i\omega_b t} H_{b+}{}^* e^{+i\omega_b t}$$
$$= \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* H_{a-} e^{-i\left(2\omega_a - \omega_b\right)t} + \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* H_{a+}{}^* e^{+i\omega_b t} \tag{64}$$
$$+\Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* H_{b-} e^{-i\omega_a t} + \Gamma_{a+} H_{a+} \Gamma_{b-}{}^* H_{b-}{}^* H_{b+}{}^* e^{+i\left(2\omega_b - \omega_a\right)t}$$

$$
\begin{aligned}
\Upsilon_6 &= \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}e^{+i\omega_d t}\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]\\
&= \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}e^{+i\omega_a t}e^{-i\omega_b t}H_{a-}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}e^{+i\omega_a t}e^{-i\omega_b t}H_{a+}{}^{*}e^{+i\omega_a t}\\
&+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}e^{+i\omega_a t}e^{-i\omega_b t}H_{b-}e^{-i\omega_b t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}e^{+i\omega_a t}e^{-i\omega_b t}H_{b+}{}^{*}e^{+i\omega_b t}\\
&= \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}H_{a-}e^{-i\omega_b t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}H_{a+}{}^{*}e^{+i\left(2\omega_a-\omega_b\right)t}\\
&+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}H_{b-}e^{-i\left(2\omega_b-\omega_a\right)t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}H_{b+}{}^{*}e^{+i\omega_a t}\\
\Upsilon_7 &= \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_s t}\left[H_{a-}e^{-i\omega_a t}+H_{a+}{}^{*}e^{+i\omega_a t}+H_{b-}e^{-i\omega_b t}+H_{b+}{}^{*}e^{+i\omega_b t}\right]\\
&= \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_a t}e^{+i\omega_b t}H_{a-}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_a t}e^{+i\omega_b t}H_{a+}{}^{*}e^{+i\omega_a t}\\
&+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_a t}e^{+i\omega_b t}H_{b-}e^{-i\omega_b t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_a t}e^{+i\omega_b t}H_{b+}{}^{*}e^{+i\omega_b t}\\
&= \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{a-}e^{+i\omega_b t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{a+}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t}\\
&+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{b-}e^{+i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{b+}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t}
\end{aligned}
\tag{65}
$$

$$\Upsilon_8 = \Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$

$$= \Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} H_{a-}e^{-i\omega_a t} + \Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} H_{a+}{}^* e^{+i\omega_a t}$$

$$+\Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} H_{b-}e^{-i\omega_b t} + \Gamma_{b+}{}^2 H_{b+}{}^2 e^{-2i\omega_b t} H_{b+}{}^* e^{+i\omega_b t}$$

$$= \Gamma_{b+}{}^2 H_{b+}{}^2 H_{a-} e^{-i\left(2\omega_b + \omega_a\right)t} + \Gamma_{b+}{}^2 H_{b+}{}^2 H_{a+}{}^* e^{-i\left(2\omega_b - \omega_a\right)t}$$

$$+\Gamma_{b+}{}^2 H_{b+}{}^2 H_{b-} e^{-3i\omega_b t} + \Gamma_{b+}{}^2 H_{b+}{}^2 H_{b+}{}^* e^{-i\omega_b t}$$

$$\Upsilon_9 = \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t}\left[H_{a-}e^{-i\omega_a t} + H_{a+}{}^* e^{+i\omega_a t} + H_{b-}e^{-i\omega_b t} + H_{b+}{}^* e^{+i\omega_b t}\right]$$

$$= \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} H_{a-}e^{-i\omega_a t} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} H_{a+}{}^* e^{+i\omega_a t}$$

$$+\Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} H_{b-}e^{-i\omega_b t} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} e^{+2i\omega_b t} H_{b+}{}^* e^{+i\omega_b t}$$

$$= \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} H_{a-} e^{+i\left(2\omega_b - \omega_a\right)t} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} H_{a+}{}^* e^{+i\left(2\omega_b + \omega_a\right)t} \tag{66}$$

$$+\Gamma_{b-}{}^{*2} H_{b-}{}^{*2} H_{b-} e^{+i\omega_b t} + \Gamma_{b-}{}^{*2} H_{b-}{}^{*2} H_{b+}{}^* e^{+3i\omega_b t}$$

$$\mathbb{S}_3 = \left|\mathcal{G}_1\right|^2 \mathcal{G}$$

$$= \left\{\begin{array}{l} \left|\Gamma_{a+}H_{a+}\right|^2 + \left|\Gamma_{a-}{}^* H_{a-}{}^*\right|^2 + 2\,\mathrm{Re}\left(\Gamma_{a-}{}^* H_{a-}{}^* \Gamma_{a+} H_{a+} e^{-2i\omega_a t}\right) \\ + \left|\Gamma_{b+}H_{b+}\right|^2 + \left|\Gamma_{b-}{}^* H_{b-}{}^*\right|^2 + 2\,\mathrm{Re}\left(\Gamma_{b-}{}^* H_{b-}{}^* \Gamma_{b+} H_{b+} e^{-2i\omega_b t}\right) \\ +2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_s t}\right) \\ +2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^* H_{b-}{}^* + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^* H_{b+}{}^*\right]e^{-i\omega_d t}\right) \end{array}\right\} \left\{\begin{array}{l} \Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^* H_{a-}{}^* e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^* H_{b-}{}^* e^{+i\omega_b t} \end{array}\right\} \tag{67}$$

$$□ \sum_{m=1}^{5} \mathrm{M}_m$$

$$\mathrm{M}_1 = \left\{\left|\Gamma_{a+}H_{a+}\right|^2 + \left|\Gamma_{a-}{}^* H_{a-}{}^*\right|^2 + \left|\Gamma_{b+}H_{b+}\right|^2 + \left|\Gamma_{b-}{}^* H_{b-}{}^*\right|^2\right\} \left\{\begin{array}{l} \Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^* H_{a-}{}^* e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^* H_{b-}{}^* e^{+i\omega_b t} \end{array}\right\} \tag{68}$$

$$
\begin{aligned}
M_2 &= 2\,\mathrm{Re}\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\right)\left\{\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}+\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\right\}\\
&=\left\{\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\right\}\left\{\begin{array}{l}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{array}\right\}\\
&=\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&\quad+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}e^{-2i\omega_a t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\\
&\quad+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&\quad+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}e^{+2i\omega_a t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\\
&=\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{a+}H_{a+}e^{-3i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{-i\omega_a t}\\
&\quad+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}e^{-i\left(2\omega_a+\omega_b\right)t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\left(2\omega_a-\omega_b\right)t}\\
&\quad+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{a+}H_{a+}e^{+i\omega_a t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+3i\omega_a t}\\
&\quad+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}e^{+i\left(2\omega_a-\omega_b\right)t}+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t}
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
M_3 &= 2\,\mathrm{Re}\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\right)\left\{\begin{array}{l}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{array}\right\}\\
&=\left\{\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\right\}\left\{\begin{array}{l}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{array}\right\}\\
&=\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&\quad+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\\
&\quad+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&\quad+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+2i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\\
&=\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{a+}H_{a+}e^{-i\left(2\omega_b+\omega_a\right)t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{-i\left(2\omega_b-\omega_a\right)t}\\
&\quad+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{b+}H_{b+}e^{-3i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\omega_b t}\\
&\quad+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{a+}H_{a+}e^{+i\left(2\omega_b-\omega_a\right)t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t}\\
&\quad+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{b+}H_{b+}e^{+i\omega_b t}+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+3i\omega_b t}
\end{aligned}
\tag{70}
$$

$$M_4 = 2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_s t}\right)\left\{\begin{array}{l}\Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{array}\right\}$$

$$= \left\{\begin{array}{l}\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t} \\ +\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\end{array}\right\}\left\{\begin{array}{l}\Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{array}\right\}$$

$$\begin{aligned}
&= \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]e^{-i\omega_a t}e^{-i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]e^{+i\omega_a t}e^{+i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}
\end{aligned} \tag{71}$$

$$\begin{aligned}
&= \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{a+}H_{a+}e^{-i\left(2\omega_a + \omega_b\right)t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{-i\omega_b t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b + \omega_a\right)t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\omega_a t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{a+}H_{a+}e^{+i\omega_b t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{b+}H_{b+}e^{+i\omega_a t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b + \omega_a\right)t}
\end{aligned} \tag{72}$$

$$
\begin{aligned}
\mathrm{M}_5 &= 2\,\mathrm{Re}\left(\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_d t}\right)\left\{\begin{matrix}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{matrix}\right\}\\
&=\left\{\begin{matrix}\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}\\ +\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\end{matrix}\right\}\left\{\begin{matrix}\Gamma_{a+}H_{a+}e^{-i\omega_a t}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t}+\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\end{matrix}\right\}\\
&=\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]e^{-i\omega_a t}e^{+i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\Gamma_{a+}H_{a+}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_a t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\Gamma_{b+}H_{b+}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]e^{+i\omega_a t}e^{-i\omega_b t}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
&=\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{a+}H_{a+}e^{-i\left(2\omega_a-\omega_b\right)t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\omega_b t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{b+}H_{b+}e^{-i\omega_a t}\\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b-\omega_a\right)t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{a+}H_{a+}e^{-i\omega_b t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_a-\omega_b\right)t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b-\omega_a\right)t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-}+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_a t}
\end{aligned}
\tag{74}
$$

and

$$\mathbb{S}_4 = \mathrm{h}_z \mathcal{G}_2 = \mathrm{Re}\left\{ h_{az} e^{-i\omega_a t} + h_{bz} e^{-i\omega_b t} \right\} \cdot \mathcal{G}_2$$

$$= \frac{1}{2} \begin{pmatrix} h_{az} e^{-i\omega_a t} + h_{bz} e^{-i\omega_b t} \\ + h_{az}{}^{*} e^{i\omega_a t} + h_{bz}{}^{*} e^{i\omega_b t} \end{pmatrix} \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} + \Gamma_0 S_0 + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} \end{bmatrix}$$

$$= \frac{1}{2} \left(\begin{array}{l} \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} + \Gamma_0 S_0 + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} \end{bmatrix} h_{az} e^{-i\omega_a t} \\ + \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} + \Gamma_0 S_0 + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} \end{bmatrix} h_{bz} e^{-i\omega_b t} \\ + \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} + \Gamma_0 S_0 + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} \end{bmatrix} h_{az}{}^{*} e^{i\omega_a t} \\ + \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} + \Gamma_0 S_0 + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} \end{bmatrix} h_{bz}{}^{*} e^{i\omega_b t} \end{array} \right) \tag{75}$$

$$\frac{1}{2} \left(\begin{array}{l} \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} h_{az} e^{-i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} h_{az} e^{-i\omega_a t} + \Gamma_0 S_0 h_{az} e^{-i\omega_a t} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} h_{az} e^{-i\omega_a t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} h_{az} e^{-i\omega_a t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} h_{az} e^{-i\omega_a t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} h_{az} e^{-i\omega_a t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} h_{az} e^{-i\omega_a t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} h_{az} e^{-i\omega_a t} \end{bmatrix} \\ + \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} h_{bz} e^{-i\omega_b t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} h_{bz} e^{-i\omega_b t} + \Gamma_0 S_0 h_{bz} e^{-i\omega_b t} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} h_{bz} e^{-i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} h_{bz} e^{-i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} h_{bz} e^{-i\omega_b t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} h_{bz} e^{-i\omega_b t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} h_{bz} e^{-i\omega_b t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} h_{bz} e^{-i\omega_b t} \end{bmatrix} \\ + \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} h_{az}{}^{*} e^{i\omega_a t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} h_{az}{}^{*} e^{i\omega_a t} + \Gamma_0 S_0 h_{az}{}^{*} e^{i\omega_a t} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} h_{az}{}^{*} e^{i\omega_a t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} h_{az}{}^{*} e^{i\omega_a t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} h_{az}{}^{*} e^{i\omega_a t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} h_{az}{}^{*} e^{i\omega_a t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} h_{az}{}^{*} e^{i\omega_a t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} h_{az}{}^{*} e^{i\omega_a t} \end{bmatrix} \\ + \begin{bmatrix} \Gamma_{2a+} S_{2a+} e^{-2i\omega_a t} h_{bz}{}^{*} e^{i\omega_b t} + \Gamma_{2a-} S_{2a-} e^{+2i\omega_a t} h_{bz}{}^{*} e^{i\omega_b t} + \Gamma_0 S_0 h_{bz}{}^{*} e^{i\omega_b t} + \Gamma_{2b+} S_{2b+} e^{-2i\omega_b t} h_{bz}{}^{*} e^{i\omega_b t} + \Gamma_{2b-} S_{2b-} e^{+2i\omega_b t} h_{bz}{}^{*} e^{i\omega_b t} \\ + \Gamma_{s+} S_{s+} e^{-i\omega_s t} h_{bz}{}^{*} e^{i\omega_b t} + \Gamma_{s-} S_{s-} e^{+i\omega_s t} h_{bz}{}^{*} e^{i\omega_b t} + \Gamma_{d+} S_{d+} e^{-i\omega_d t} h_{bz}{}^{*} e^{i\omega_b t} + \Gamma_{d-} S_{d-} e^{+i\omega_d t} h_{bz}{}^{*} e^{i\omega_b t} \end{bmatrix} \end{array} \right) \tag{76}$$

$$
=\frac{1}{2}\left(\begin{aligned}
&\left[\begin{aligned}&\Gamma_{2a+}S_{2a+}h_{az}e^{-3i\omega_a t}+\Gamma_{2a-}S_{2a-}h_{az}e^{+i\omega_a t}+\Gamma_0 S_0 h_{az}e^{-i\omega_a t}+\boxed{\Gamma_{2b+}S_{2b+}h_{az}e^{-i\left(2\omega_b+\omega_a\right)t}}+\boxed{\Gamma_{2b-}S_{2b-}h_{az}e^{+i\left(2\omega_b-\omega_a\right)t}}\\
&+\boxed{\Gamma_{s+}S_{s+}h_{az}e^{-i\left(2\omega_a+\omega_b\right)t}}+\Gamma_{s-}S_{s-}h_{az}e^{+i\omega_b t}+\boxed{\Gamma_{d+}S_{d+}h_{az}e^{-i\left(2\omega_a-\omega_b\right)t}}+\Gamma_{d-}S_{d-}h_{az}e^{-i\omega_b t}\end{aligned}\right]\\
&+\left[\begin{aligned}&\boxed{\Gamma_{2a+}S_{2a+}h_{bz}e^{-i\left(2\omega_a+\omega_b\right)t}}+\boxed{\Gamma_{2a-}S_{2a-}h_{bz}e^{+i\left(2\omega_a-\omega_b\right)t}}+\Gamma_0 S_0 h_{bz}e^{-i\omega_b t}+\Gamma_{2b+}S_{2b+}h_{bz}e^{-3i\omega_b t}+\Gamma_{2b-}S_{2b-}h_{bz}e^{+i\omega_b t}\\
&+\boxed{\Gamma_{s+}S_{s+}h_{bz}e^{-i\left(2\omega_b+\omega_a\right)t}}+\Gamma_{s-}S_{s-}h_{bz}e^{+i\omega_a t}+\Gamma_{d+}S_{d+}h_{bz}e^{-i\omega_a t}+\boxed{\Gamma_{d-}S_{d-}h_{bz}e^{-i\left(2\omega_b-\omega_a\right)t}}\end{aligned}\right]\\
&+\left[\begin{aligned}&\Gamma_{2a+}S_{2a+}h_{az}^{*}e^{-i\omega_a t}+\Gamma_{2a-}S_{2a-}h_{az}^{*}e^{3i\omega_a t}+\Gamma_0 S_0 h_{az}^{*}e^{i\omega_a t}+\boxed{\Gamma_{2b+}S_{2b+}h_{az}^{*}e^{-i\left(2\omega_b-\omega_a\right)t}}+\boxed{\Gamma_{2b-}S_{2b-}h_{az}^{*}e^{+i\left(2\omega_b+\omega_a\right)t}}\\
&+\Gamma_{s+}S_{s+}h_{az}^{*}e^{-i\omega_b t}+\boxed{\Gamma_{s-}S_{s-}h_{az}^{*}e^{+i\left(2\omega_a+\omega_b\right)t}}+\Gamma_{d+}S_{d+}h_{az}^{*}e^{i\omega_b t}+\boxed{\Gamma_{d-}S_{d-}h_{az}^{*}e^{+i\left(2\omega_a-\omega_b\right)t}}\end{aligned}\right]\\
&+\left[\begin{aligned}&\boxed{\Gamma_{2a+}S_{2a+}h_{bz}^{*}e^{-i\left(2\omega_a-\omega_b\right)t}}+\boxed{\Gamma_{2a-}S_{2a-}h_{bz}^{*}e^{+i\left(2\omega_a+\omega_b\right)t}}+\Gamma_0 S_0 h_{bz}^{*}e^{i\omega_b t}+\Gamma_{2b+}S_{2b+}h_{bz}^{*}e^{-i\omega_b t}+\Gamma_{2b-}S_{2b-}h_{bz}^{*}e^{3i\omega_b t}\\
&+\Gamma_{s+}S_{s+}h_{bz}^{*}e^{-i\omega_a t}+\boxed{\Gamma_{s-}S_{s-}h_{bz}^{*}e^{+i\left(2\omega_b+\omega_a\right)t}}+\boxed{\Gamma_{d+}S_{d+}h_{bz}^{*}e^{+i\left(2\omega_b-\omega_a\right)t}}+\Gamma_{d-}S_{d-}h_{bz}^{*}e^{+i\omega_a t}\end{aligned}\right]
\end{aligned}\right) \tag{77}
$$

$$
=\frac{1}{2}\left(\begin{aligned}
&\Gamma_{2a+}S_{2a+}h_{az}e^{-3i\omega_a t}+\Gamma_{2b+}S_{2b+}h_{bz}e^{-3i\omega_b t}+\Gamma_{2a-}S_{2a-}h_{az}^{*}e^{3i\omega_a t}+\Gamma_{2b-}S_{2b-}h_{bz}^{*}e^{3i\omega_b t}\\
&+\left[\Gamma_{2a-}S_{2a-}h_{az}+\Gamma_{s-}S_{s-}h_{bz}+\Gamma_{d-}S_{d-}h_{bz}^{*}+\Gamma_0 S_0 h_{az}^{*}\right]e^{+i\omega_a t}\\
&+\left[\Gamma_0 S_0 h_{az}+\Gamma_{d+}S_{d+}h_{bz}+\Gamma_{s+}S_{s+}h_{bz}^{*}+\Gamma_{2a+}S_{2a+}h_{az}^{*}\right]e^{-i\omega_a t}\\
&+\left[\Gamma_{d+}S_{d+}h_{az}^{*}+\Gamma_0 S_0 h_{bz}^{*}+\Gamma_{2b-}S_{2b-}h_{bz}+\Gamma_{s-}S_{s-}h_{az}\right]e^{+i\omega_b t}\\
&+\left[\Gamma_{d-}S_{d-}h_{az}+\Gamma_0 S_0 h_{bz}+\Gamma_{2b+}S_{2b+}h_{bz}^{*}+\Gamma_{s+}S_{s+}h_{az}^{*}e^{-i\omega_b t}\right]e^{-i\omega_b t}\\
&+\left[\Gamma_{2b+}S_{2b+}h_{az}+\Gamma_{s+}S_{s+}h_{bz}\right]e^{-i\left(2\omega_b+\omega_a\right)t}+\left[\Gamma_{2b-}S_{2b-}h_{az}^{*}+\Gamma_{s-}S_{s-}h_{bz}^{*}\right]e^{+i\left(2\omega_b+\omega_a\right)t}\\
&+\left[\Gamma_{s+}S_{s+}h_{az}+\Gamma_{2a+}S_{2a+}h_{bz}\right]e^{-i\left(2\omega_a+\omega_b\right)t}+\left[\Gamma_{s-}S_{s-}h_{az}^{*}+\Gamma_{2a-}S_{2a-}h_{bz}^{*}\right]e^{+i\left(2\omega_a+\omega_b\right)t}\\
&+\left[\Gamma_{d+}S_{d+}h_{az}+\Gamma_{2a+}S_{2a+}h_{bz}^{*}\right]e^{-i\left(2\omega_a-\omega_b\right)t}+\left[\Gamma_{2a-}S_{2a-}h_{bz}+\Gamma_{d-}S_{d-}h_{az}^{*}\right]e^{+i\left(2\omega_a-\omega_b\right)t}\\
&+\left[\Gamma_{d-}S_{d-}h_{bz}+\Gamma_{2b+}S_{2b+}h_{az}^{*}\right]e^{-i\left(2\omega_b-\omega_a\right)t}+\left[\Gamma_{2b-}S_{2b-}h_{az}+\Gamma_{d+}S_{d+}h_{bz}^{*}\right]e^{+i\left(2\omega_b-\omega_a\right)t}
\end{aligned}\right) \tag{78}
$$

$$
\mathbb{S}_1\Big|_{\mathrm{IM}} = \frac{1}{2}\left\{
\begin{aligned}
&\boxed{\Gamma_{a-}^{*} H_{a-}^{*} \Gamma_{a+} H_{a+} H_{b+}}\, e^{-i\left(2\omega_a + \omega_b\right)t} + \boxed{\Gamma_{a-}^{*} H_{a-}^{*} \Gamma_{a+} H_{a+} H_{b-}^{*}}\, e^{-i\left(2\omega_a - \omega_b\right)t} \\
&+ \boxed{\Gamma_{a-} H_{a-} \Gamma_{a+}^{*} H_{a+}^{*} H_{b+}}\, e^{+i\left(2\omega_a - \omega_b\right)t} + \boxed{\Gamma_{a-} H_{a-} \Gamma_{a+}^{*} H_{a+}^{*} H_{b-}^{*}}\, e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+ \boxed{\Gamma_{b-}^{*} H_{b-}^{*} \Gamma_{b+} H_{b+} H_{a+}}\, e^{-i\left(2\omega_b + \omega_a\right)t} + \boxed{\Gamma_{b-}^{*} H_{b-}^{*} \Gamma_{b+} H_{b+} H_{a-}^{*}}\, e^{-i\left(2\omega_b - \omega_a\right)t} \\
&+ \boxed{\Gamma_{b-} H_{b-} \Gamma_{b+}^{*} H_{b+}^{*} H_{a+}}\, e^{+i\left(2\omega_b - \omega_a\right)t} + \boxed{\Gamma_{b-} H_{b-} \Gamma_{b+}^{*} H_{b+}^{*} H_{a-}^{*}}\, e^{+i\left(2\omega_b + \omega_a\right)t} \\
&+ \boxed{\left[\Gamma_{a-} H_{a-} \Gamma_{b+} H_{b+} + \Gamma_{a+} H_{a+} \Gamma_{b-} H_{b-}\right] H_{a+}}\, e^{-i\left(2\omega_a + \omega_b\right)t} \\
&+ \boxed{\left[\Gamma_{a-} H_{a-} \Gamma_{b+} H_{b+} + \Gamma_{a+} H_{a+} \Gamma_{b-} H_{b-}\right] H_{b+}}\, e^{-i\left(2\omega_b + \omega_a\right)t} \\
&+ \boxed{\left[\Gamma_{a-}^{*} H_{a-}^{*} \Gamma_{b+}^{*} H_{b+}^{*} + \Gamma_{a+}^{*} H_{a+}^{*} \Gamma_{b-}^{*} H_{b-}^{*}\right] H_{a-}^{*}}\, e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+ \boxed{\left[\Gamma_{a-}^{*} H_{a-}^{*} \Gamma_{b+}^{*} H_{b+}^{*} + \Gamma_{a+}^{*} H_{a+}^{*} \Gamma_{b-}^{*} H_{b-}^{*}\right] H_{b-}^{*}}\, e^{+i\left(2\omega_b + \omega_a\right)t} \\
&+ \boxed{\left[\Gamma_{a-} H_{a-} \Gamma_{b-}^{*} H_{b-}^{*} + \Gamma_{a+} H_{a+} \Gamma_{b+}^{*} H_{b+}^{*}\right] H_{a+}}\, e^{-i\left(2\omega_a - \omega_b\right)t} \\
&+ \boxed{\left[\Gamma_{a-} H_{a-} \Gamma_{b-}^{*} H_{b-}^{*} + \Gamma_{a+} H_{a+} \Gamma_{b+}^{*} H_{b+}^{*}\right] H_{b-}^{*}}\, e^{+i\left(2\omega_b - \omega_a\right)t} \\
&+ \boxed{\left[\Gamma_{a-}^{*} H_{a-}^{*} \Gamma_{b-} H_{b-} + \Gamma_{a+}^{*} H_{a+}^{*} \Gamma_{b+} H_{b+}\right] H_{a-}^{*}}\, e^{+i\left(2\omega_a - \omega_b\right)t} \\
&+ \boxed{\left[\Gamma_{a-}^{*} H_{a-}^{*} \Gamma_{b-} H_{b-} + \Gamma_{a+}^{*} H_{a+}^{*} \Gamma_{b+} H_{b+}\right] H_{b+}}\, e^{-i\left(2\omega_b - \omega_a\right)t}
\end{aligned}
\right\} \qquad (79)
$$

$$
= \frac{1}{2}\left\{
\begin{aligned}
&\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{b+} + \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{a+}\right)e^{-i\left(2\omega_a + \omega_b\right)t} \\
&+\left(\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{b-}{}^{*} + \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{a-}{}^{*}\right)e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+\left(\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}H_{b-}{}^{*} + \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{a+}\right)e^{-i\left(2\omega_a - \omega_b\right)t} \\
&+\left(\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}H_{b+} + \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{a-}{}^{*}\right)e^{+i\left(2\omega_a - \omega_b\right)t} \\
&+\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{a+} + \left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]H_{b+}\right)e^{-i\left(2\omega_b + \omega_a\right)t} \\
&+\left(\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{a-}{}^{*} + \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]H_{b-}{}^{*}\right)e^{+i\left(2\omega_b + \omega_a\right)t} \\
&+\left(\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}H_{a-}{}^{*} + \left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]H_{b+}\right)e^{-i\left(2\omega_b - \omega_a\right)t} \\
&+\left(\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}H_{a+} + \left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]H_{b-}{}^{*}\right)e^{+i\left(2\omega_b - \omega_a\right)t}
\end{aligned}
\right\} \quad (80)
$$

$$
\mathbb{S}_2\Big|_{\mathrm{IM}} = \frac{1}{2}\left\{
\begin{aligned}
&\Gamma_{a+}{}^{2}H_{a+}{}^{2}H_{b+}e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{a+}{}^{2}H_{a+}{}^{2}H_{b-}{}^{*}e^{-i\left(2\omega_a - \omega_b\right)t} \\
&+\Gamma_{a-}{}^{*2}H_{a-}{}^{*2}H_{b+}e^{+i\left(2\omega_a - \omega_b\right)t} + \Gamma_{a-}{}^{*2}H_{a-}{}^{*2}H_{b-}{}^{*}e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}H_{a+}e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}H_{b+}e^{-i\left(2\omega_b + \omega_a\right)t} \\
&+\Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{a+}e^{-i\left(2\omega_a - \omega_b\right)t} + \Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b - \omega_a\right)t} \\
&+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}H_{a-}{}^{*}e^{+i\left(2\omega_a - \omega_b\right)t} + \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}H_{b+}e^{-i\left(2\omega_b - \omega_a\right)t} \\
&+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_a + \omega_b\right)t} + \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b + \omega_a\right)t} \\
&+\Gamma_{b+}{}^{2}H_{b+}{}^{2}H_{a+}e^{-i\left(2\omega_b + \omega_a\right)t} + \Gamma_{b+}{}^{2}H_{b+}{}^{2}H_{a-}{}^{*}e^{-i\left(2\omega_b - \omega_a\right)t} \\
&+\Gamma_{b-}{}^{*2}H_{b-}{}^{*2}H_{a+}{}^{+i\left(2\omega_b - \omega_a\right)t} + \Gamma_{b-}{}^{*2}H_{b-}{}^{*2}H_{a-}{}^{*}e^{+i\left(2\omega_b + \omega_a\right)t}
\end{aligned}
\right\} \quad (81)
$$

$$
=\frac{1}{2}\left\{\begin{aligned}
&\left[\Gamma_{a+}{}^{2}H_{a+}{}^{2}H_{b+}+\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}H_{a+}\right]e^{-i\left(2\omega_a+\omega_b\right)t}\\
&+\left[\Gamma_{a+}{}^{2}H_{a+}{}^{2}H_{b-}{}^{*}+\Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{a+}\right]e^{-i\left(2\omega_a-\omega_b\right)t}\\
&+\left[\Gamma_{a-}{}^{*2}H_{a-}{}^{*2}H_{b+}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}H_{a-}{}^{*}\right]e^{+i\left(2\omega_a-\omega_b\right)t}\\
&+\left[\Gamma_{a-}{}^{*2}H_{a-}{}^{*2}H_{b-}{}^{*}+\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{a-}{}^{*}\right]e^{+i\left(2\omega_a+\omega_b\right)t}\\
&+\left[\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}H_{b+}+\Gamma_{b+}{}^{2}H_{b+}{}^{2}H_{a+}\right]e^{-i\left(2\omega_b+\omega_a\right)t}\\
&+\left[\Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{b-}{}^{*2}H_{b-}{}^{*2}H_{a+}\right]e^{+i\left(2\omega_b-\omega_a\right)t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}H_{b+}H_{b+}+\Gamma_{b+}{}^{2}H_{b+}{}^{2}H_{a-}{}^{*}\right]e^{-i\left(2\omega_b-\omega_a\right)t}\\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}H_{b-}{}^{*}+\Gamma_{b-}{}^{*2}H_{b-}{}^{*2}H_{a-}{}^{*}\right]e^{+i\left(2\omega_b+\omega_a\right)t}
\end{aligned}\right\}\tag{82}
$$

$$
\begin{aligned}
\mathbb{S}_3\Big|_{\mathrm{IM}} &= \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{b+}H_{b+}e^{-i\left(2\omega_a+\omega_b\right)t} + \Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{a+}H_{a+}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\left(2\omega_a-\omega_b\right)t} \\
&+\Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}e^{+i\left(2\omega_a-\omega_b\right)t} + \Gamma_{a-}H_{a-}\Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t} \\
&+\Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{a+}H_{a+}e^{-i\left(2\omega_b+\omega_a\right)t} + \Gamma_{b-}{}^{*}H_{b-}{}^{*}\Gamma_{b+}H_{b+}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{-i\left(2\omega_b-\omega_a\right)t} \\
&+\Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{a+}H_{a+}e^{+i\left(2\omega_b-\omega_a\right)t} + \Gamma_{b-}H_{b-}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{a+}H_{a+}e^{-i\left(2\omega_a+\omega_b\right)t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b+}H_{b+} + \Gamma_{a+}H_{a+}\Gamma_{b-}H_{b-}\right]\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b+\omega_a\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_a+\omega_b\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b+}{}^{*}H_{b+}{}^{*} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b+\omega_a\right)t} \\
&\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{a+}H_{a+}e^{-i\left(2\omega_a-\omega_b\right)t} \\
&+\left[\Gamma_{a-}H_{a-}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + \Gamma_{a+}H_{a+}\Gamma_{b+}{}^{*}H_{b+}{}^{*}\right]\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b-\omega_a\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{a-}{}^{*}H_{a-}{}^{*}e^{+i\left(2\omega_a-\omega_b\right)t} \\
&+\left[\Gamma_{a-}{}^{*}H_{a-}{}^{*}\Gamma_{b-}H_{b-} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\Gamma_{b+}H_{b+}\right]\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b-\omega_a\right)t}
\end{aligned}
\qquad (83)
$$

$$
\begin{aligned}
&= \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+} H_{b+} + \Gamma_{a-} H_{a-} \Gamma_{b+} H_{b+} + \Gamma_{a+} H_{a+} \Gamma_{b-} H_{b-} \right] \Gamma_{a+} H_{a+} e^{-i\left(2\omega_a + \omega_b\right)t} \\
&+ \left[\Gamma_{a-} H_{a-} \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+}{}^{*} H_{b+}{}^{*} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} \right] \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right] e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+ \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + \Gamma_{a-} H_{a-} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + \Gamma_{a+} H_{a+} \Gamma_{b+}{}^{*} H_{b+}{}^{*} \right] \Gamma_{a+} H_{a+} e^{-i\left(2\omega_a - \omega_b\right)t} \\
&+ \left[\Gamma_{a-} H_{a-} \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b+} H_{b+} + \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-} H_{b-} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b+} H_{b+} \right] \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right] e^{+i\left(2\omega_a - \omega_b\right)t} \\
&+ \left[\Gamma_{b-}{}^{*} H_{b-}{}^{*} \Gamma_{a+} H_{a+} + \Gamma_{a-} H_{a-} \Gamma_{b+} H_{b+} + \Gamma_{a+} H_{a+} \Gamma_{b-} H_{b-} \right] \Gamma_{b+} H_{b+} e^{-i\left(2\omega_b + \omega_a\right)t} \\
&+ \left[\Gamma_{b-} H_{b-} \Gamma_{b+}{}^{*} H_{b+}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + \left[\Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b+}{}^{*} H_{b+}{}^{*} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} \right] \Gamma_{b-}{}^{*} H_{b-}{}^{*} \right] e^{+i\left(2\omega_b + \omega_a\right)t} \\
&+ \left[\Gamma_{b-}{}^{*} H_{b-}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + \Gamma_{a-}{}^{*} H_{a-}{}^{*} \Gamma_{b-} H_{b-} + \Gamma_{a+}{}^{*} H_{a+}{}^{*} \Gamma_{b+} H_{b+} \right] \Gamma_{b+} H_{b+} e^{-i\left(2\omega_b - \omega_a\right)t} \\
&+ \left[\Gamma_{b-} H_{b-} \Gamma_{b+}{}^{*} H_{b+}{}^{*} \Gamma_{a+} H_{a+} + \left[\Gamma_{a-} H_{a-} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + \Gamma_{a+} H_{a+} \Gamma_{b+}{}^{*} H_{b+}{}^{*} \right] \Gamma_{b-}{}^{*} H_{b-}{}^{*} \right] e^{+i\left(2\omega_b - \omega_a\right)t}
\end{aligned}
$$

$$
\mathbb{S}_4\Big|_{\mathrm{IM}} = \frac{1}{2}\begin{pmatrix}
+\left[\Gamma_{2b+} S_{2b+} h_{az} + \Gamma_{sum+} S_{sum+} h_{bz} \right] e^{-i\left(2\omega_b + \omega_a\right)t} + \left[\Gamma_{2b-} S_{2b-} h_{az}{}^{*} + \Gamma_{sum-} S_{sum-} h_{bz}{}^{*} \right] e^{+i\left(2\omega_b + \omega_a\right)t} \\
+\left[\Gamma_{2a+} S_{2a+} h_{bz} + \Gamma_{sum+} S_{sum+} h_{az} + \right] e^{-i\left(2\omega_a + \omega_b\right)t} + \left[\Gamma_{2a-} S_{2a-} h_{bz}{}^{*} + \Gamma_{sum-} S_{sum-} h_{az}{}^{*} \right] e^{+i\left(2\omega_a + \omega_b\right)t} \\
+\left[\Gamma_{2a+} S_{2a+} h_{bz}{}^{*} + \Gamma_{dif+} S_{dif+} h_{az} \right] e^{-i\left(2\omega_a - \omega_b\right)t} + \left[\Gamma_{2a-} S_{2a-} h_{bz} + \Gamma_{dif-} S_{dif-} h_{az}{}^{*} \right] e^{+i\left(2\omega_a - \omega_b\right)t} \\
+\left[\Gamma_{2b+} S_{2b+} h_{az}{}^{*} + \Gamma_{dif-} S_{dif-} h_{bz} \right] e^{-i\left(2\omega_b - \omega_a\right)t} + \left[\Gamma_{2b-} S_{2b-} h_{az} + \Gamma_{dif+} S_{dif+} h_{bz}{}^{*} \right] e^{+i\left(2\omega_b - \omega_a\right)t}
\end{pmatrix} \quad (84)
$$

$$
\begin{bmatrix}
S_{2a+}e^{-2i\omega_a t}+S_{2a-}e^{+2i\omega_a t}+S_0+S_{2b+}e^{-2i\omega_b t}+S_{2b-}e^{+2i\omega_b t}\\
+S_{sum+}e^{-i\omega_{sum}t}+S_{sum-}e^{+i\omega_{sum}t}+S_{dif+}e^{-i\omega_{dif}t}+S_{dif-}e^{+i\omega_{dif}t}
\end{bmatrix}
$$

$$
=\left\{
\begin{array}{l}
\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-2i\omega_a t}+\left(h_{az}{}^*\Gamma_{a-}{}^*H_{a-}{}^*\right)e^{+2i\omega_a t}\\
+h_{az}\Gamma_{a-}{}^*H_{a-}{}^*+h_{bz}\Gamma_{b-}{}^*H_{b-}{}^*+h_{az}{}^*\Gamma_{a+}H_{a+}+h_{bz}{}^*\Gamma_{b+}H_{b+}\\
+h_{bz}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}+h_{bz}{}^*\Gamma_{b-}{}^*H_{b-}{}^*e^{+2i\omega_b t}\\
+\left(h_{az}\Gamma_{b+}H_{b+}+h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_{sum}t}+\left(h_{az}{}^*\Gamma_{b-}{}^*H_{b-}{}^*+h_{bz}{}^*\Gamma_{a-}{}^*H_{a-}{}^*\right)e^{+i\omega_{sum}t}\\
+\left(h_{az}\Gamma_{b-}{}^*H_{b-}{}^*+h_{bz}{}^*\Gamma_{a+}H_{a+}\right)e^{-i\omega_{dif}t}+\left(h_{bz}\Gamma_{a-}{}^*H_{a-}{}^*+h_{az}{}^*\Gamma_{b+}H_{b+}\right)e^{+i\omega_{dif}t}
\end{array}
\right\}
$$

$$
\Rightarrow\left\{
\begin{array}{c}
S_{2a+}=\left(h_{az}\Gamma_{a+}H_{a+}\right)\\
S_{2a-}=\left(h_{az}{}^*\Gamma_{a-}{}^*H_{a-}{}^*\right)\\
S_0=\left(h_{az}\Gamma_{a-}{}^*H_{a-}{}^*+h_{bz}\Gamma_{b-}{}^*H_{b-}{}^*+h_{az}{}^*\Gamma_{a+}H_{a+}+h_{bz}{}^*\Gamma_{b+}H_{b+}\right)\\
S_{2b+}=\left(h_{bz}\Gamma_{b+}H_{b+}\right)\\
S_{2b-}=\left(h_{bz}{}^*\Gamma_{b-}{}^*H_{b-}{}^*\right)\\
S_{sum+}=\left(h_{az}\Gamma_{b+}H_{b+}+h_{bz}\Gamma_{a+}H_{a+}\right)\\
S_{sum-}=\left(h_{az}{}^*\Gamma_{b-}{}^*H_{b-}{}^*+h_{bz}{}^*\Gamma_{a-}{}^*H_{a-}{}^*\right)\\
S_{dif+}=\left(h_{az}\Gamma_{b-}{}^*H_{b-}{}^*+h_{bz}{}^*\Gamma_{a+}H_{a+}\right)\\
S_{dif-}=\left(h_{bz}\Gamma_{a-}{}^*H_{a-}{}^*+h_{az}{}^*\Gamma_{b+}H_{b+}\right)
\end{array}
\right.
\qquad (85)
$$

$$
\Rightarrow \mathbb{S}_4\Big|_{\mathrm{IM}} = \frac{1}{2}\left(\begin{array}{l}
\left[\Gamma_{sum+}\left(h_{bz}\Gamma_{a+}H_{a+} + h_{az}\Gamma_{b+}H_{b+}\right)h_{bz} + \Gamma_{2b+}\left(h_{bz}\Gamma_{b+}H_{b+}\right)h_{az}\right]e^{-i\left(2\omega_b+\omega_a\right)t} \\
+\left[\Gamma_{sum-}\left(h_{bz}^*\Gamma_{a-}^*H_{a-}^* + h_{az}^*\Gamma_{b-}^*H_{b-}^*\right)h_{bz}^* + \Gamma_{2b-}\left(h_{bz}^*\Gamma_{b-}^*H_{b-}^*\right)h_{az}^*\right]e^{+i\left(2\omega_b+\omega_a\right)t} \\
+\left[\Gamma_{sum+}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)h_{az} + \Gamma_{2a+}\left(h_{az}\Gamma_{a+}H_{a+}\right)h_{bz}\right]e^{-i\left(2\omega_a+\omega_b\right)t} \\
+\left[\Gamma_{sum-}\left(h_{az}^*\Gamma_{b-}^*H_{b-}^* + h_{bz}^*\Gamma_{a-}^*H_{a-}^*\right)h_{az}^* + \Gamma_{2a-}\left(h_{az}^*\Gamma_{a-}^*H_{a-}^*\right)h_{bz}^*\right]e^{+i\left(2\omega_a+\omega_b\right)t} \\
+\left[\Gamma_{dif+}\left(h_{az}\Gamma_{b-}^*H_{b-}^* + h_{bz}^*\Gamma_{a+}H_{a+}\right)h_{az} + \Gamma_{2a+}\left(h_{az}\Gamma_{a+}H_{a+}\right)h_{bz}^*\right]e^{-i\left(2\omega_a-\omega_b\right)t} \\
+\left[\Gamma_{dif-}\left(h_{az}^*\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a-}^*H_{a-}^*\right)h_{az}^* + \Gamma_{2a-}\left(h_{az}^*\Gamma_{a-}^*H_{a-}^*\right)h_{bz}\right]e^{+i\left(2\omega_a-\omega_b\right)t} \\
+\left[\Gamma_{dif-}\left(h_{bz}\Gamma_{a-}^*H_{a-}^* + h_{az}^*\Gamma_{b+}H_{b+}\right)h_{bz} + \Gamma_{2b+}\left(h_{bz}\Gamma_{b+}H_{b+}\right)h_{az}^*\right]e^{-i\left(2\omega_b-\omega_a\right)t} \\
+\left[\Gamma_{dif+}\left(h_{bz}^*\Gamma_{a+}H_{a+} + h_{az}\Gamma_{b-}^*H_{b-}^*\right)h_{bz}^* + \Gamma_{2b-}\left(h_{bz}^*\Gamma_{b-}^*H_{b-}^*\right)h_{az}\right]e^{+i\left(2\omega_b-\omega_a\right)t}
\end{array}\right) \tag{86}
$$

For the z-component, we have

$$
\begin{aligned}
\mathrm{g}_{z3} &= -\frac{1}{2}\left(\mathcal{G}_1^*\mathcal{G}_2 + \mathcal{G}_1\mathcal{G}_2^*\right) = -\mathrm{Re}\left(\mathcal{G}_1^*\mathcal{G}_2\right) \\
&= +\mathrm{Re}\left(\left\{\begin{array}{l}\Gamma_{a+}H_{a+}e^{-i\omega_a t} + \Gamma_{a-}^*H_{a-}^*e^{+i\omega_a t} \\ +\Gamma_{b+}H_{b+}e^{-i\omega_b t} + \Gamma_{b-}^*H_{b-}^*e^{+i\omega_b t}\end{array}\right\}^* \mathcal{G}_2\right) = +\mathrm{Re}\left(\begin{array}{l}\Gamma_{a+}^*H_{a+}^*e^{+i\omega_a t}\mathcal{G}_2 + \Gamma_{a-}H_{a-}e^{-i\omega_a t}\mathcal{G}_2 \\ +\Gamma_{b+}^*H_{b+}^*e^{+i\omega_b t}\mathcal{G}_2 + \Gamma_{b-}H_{b-}e^{-i\omega_b t}\mathcal{G}_2\end{array}\right) \\
&= \mathrm{Re}\sum_{m=1}^{4}\Xi_m
\end{aligned} \tag{87}
$$

$$
\Xi_1 = \Gamma_{a+}{}^{*} H_{a+}{}^{*} e^{+i\omega_a t}
\begin{bmatrix}
\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-2i\omega_a t} + \left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+2i\omega_a t} \\
+h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+} \\
+\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_s t} + \left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_s t} \\
+\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_d t} + \left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_d t} \\
+h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+2i\omega_b t} + h_{bz}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}
\end{bmatrix}
$$

$$
\begin{aligned}
&= \Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{+i\omega_a t}e^{-2i\omega_a t} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_a t}e^{+2i\omega_a t} \\
&+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_a t} \\
&+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{+i\omega_a t}e^{-i\omega_s t} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_a t}e^{+i\omega_s t} \\
&+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{+i\omega_a t}e^{-i\omega_d t} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_a t}e^{+i\omega_d t} \\
&+\Gamma_{a+}{}^{*}H_{a+}{}^{*}h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_a t}e^{+2i\omega_b t} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}h_{bz}\Gamma_{b+}H_{b+}e^{+i\omega_a t}e^{-2i\omega_b t}
\end{aligned}
$$

$$
\begin{aligned}
&= \Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-i\omega_a t} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+3i\omega_a t} \\
&+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_a t} \\
&+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_b t} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+\Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{+i\omega_b t} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\left(2\omega_a - \omega_b\right)t} \\
&+\Gamma_{a+}{}^{*}H_{a+}{}^{*}h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b + \omega_a\right)t} + \Gamma_{a+}{}^{*}H_{a+}{}^{*}h_{bz}\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b - \omega_a\right)t}
\end{aligned}
\tag{88}
$$

$$
\Xi_2 = \Gamma_{a-}H_{a-}e^{-i\omega_a t}\left[\begin{array}{l}
\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-2i\omega_a t} + \left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+2i\omega_a t} \\
+h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+} \\
+\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_s t} + \left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_s t} \\
+\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_d t} + \left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_d t} \\
+h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+2i\omega_b t} + h_{bz}\Gamma_{b+}H_{b+}e^{-2i\omega_b t}
\end{array}\right]
$$

$$
\begin{aligned}
&= \Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-i\omega_a t}e^{-2i\omega_a t} + \Gamma_{a-}H_{a-}\left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{-i\omega_a t}e^{+2i\omega_a t} \\
&+\Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+}\right)e^{-i\omega_a t} \\
&+\Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_a t}e^{-i\omega_s t} + \Gamma_{a-}H_{a-}\left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{-i\omega_a t}e^{+i\omega_s t} \\
&+\Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_a t}e^{-i\omega_d t} + \Gamma_{a-}H_{a-}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{-i\omega_a t}e^{+i\omega_d t} \\
&+\Gamma_{a-}H_{a-}h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{-i\omega_a t}e^{+2i\omega_b t} + \Gamma_{a-}H_{a-}h_{bz}\Gamma_{b+}H_{b+}e^{-i\omega_a t}e^{-2i\omega_b t}
\end{aligned}
$$

$$
\begin{aligned}
&= \Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-3i\omega_a t} + \Gamma_{a-}H_{a-}\left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_a t} \\
&+\Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+}\right)e^{-i\omega_a t} \\
&+\Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{a-}H_{a-}\left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_b t} \\
&+\Gamma_{a-}H_{a-}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\left(2\omega_a - \omega_b\right)t} + \Gamma_{a-}H_{a-}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{-i\omega_b t} \\
&+\Gamma_{a-}H_{a-}h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\left(2\omega_b - \omega_a\right)t} + \Gamma_{a-}H_{a-}h_{bz}\Gamma_{b+}H_{b+}e^{-i\left(2\omega_b + \omega_a\right)t}
\end{aligned}
\tag{89}
$$

$$\Xi_3 = \Gamma_{b+}{}^{*}H_{b+}{}^{*}e^{+i\omega_b t}\begin{bmatrix} \left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-2i\omega_a t} + \left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+2i\omega_a t} \\ + h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+} \\ + \left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_s t} + \left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_s t} \\ + \left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{-i\omega_d t} + \left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_d t} \\ + h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+2i\omega_b t} + h_{bz}\Gamma_{b+}H_{b+}e^{-2i\omega_b t} \end{bmatrix}$$

$$\begin{aligned}
&= \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{+i\omega_b t}e^{-2i\omega_a t} + \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_b t}e^{+2i\omega_a t} \\
&+ \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_b t} \\
&+ \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{+i\omega_b t}e^{-i\omega_s t} + \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\omega_b t}e^{+i\omega_s t} \\
&+ \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{+i\omega_b t}e^{-i\omega_d t} + \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_b t}e^{+i\omega_d t} \\
&+ \Gamma_{b+}{}^{*}H_{b+}{}^{*}h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+i\omega_b t}e^{+2i\omega_b t} + \Gamma_{b+}{}^{*}H_{b+}{}^{*}h_{bz}\Gamma_{b+}H_{b+}e^{+i\omega_b t}e^{-2i\omega_b t}
\end{aligned}$$

$$\begin{aligned}
&= \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}\Gamma_{a+}H_{a+}\right)e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+ \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{bz}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{az}{}^{*}\Gamma_{a+}H_{a+} + h_{bz}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_b t} \\
&+ \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}\Gamma_{b+}H_{b+} + h_{bz}\Gamma_{a+}H_{a+}\right)e^{-i\omega_a t} + \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a-}{}^{*}H_{a-}{}^{*}\right)e^{+i\left(2\omega_b + \omega_a\right)t} \\
&+ \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{az}\Gamma_{b-}{}^{*}H_{b-}{}^{*} + h_{bz}{}^{*}\Gamma_{a+}H_{a+}\right)e^{+i\left(2\omega_b - \omega_a\right)t} + \Gamma_{b+}{}^{*}H_{b+}{}^{*}\left(h_{bz}\Gamma_{a-}{}^{*}H_{a-}{}^{*} + h_{az}{}^{*}\Gamma_{b+}H_{b+}\right)e^{+i\omega_a t} \\
&+ \Gamma_{b+}{}^{*}H_{b+}{}^{*}h_{bz}{}^{*}\Gamma_{b-}{}^{*}H_{b-}{}^{*}e^{+3i\omega_b t} + \Gamma_{b+}{}^{*}H_{b+}{}^{*}h_{bz}\Gamma_{b+}H_{b+}e^{-i\omega_b t}
\end{aligned} \tag{90}$$

$$
\Xi_4 = \Gamma_{b-} H_{b-} e^{-i\omega_b t} \left[\begin{array}{l}
\left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-2i\omega_a t} + \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+2i\omega_a t} \\
+ h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \\
+ \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\omega_s t} + \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\omega_s t} \\
+ \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{-i\omega_d t} + \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{+i\omega_d t} \\
+ h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+2i\omega_b t} + h_{bz} \Gamma_{b+} H_{b+} e^{-2i\omega_b t}
\end{array} \right]
$$

$$
\begin{aligned}
&= \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-i\omega_b t} e^{-2i\omega_a t} + \Gamma_{b-} H_{b-} \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{-i\omega_b t} e^{+2i\omega_a t} \\
&+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \right) e^{-i\omega_b t} \\
&+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\omega_b t} e^{-i\omega_s t} + \Gamma_{b-} H_{b-} \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{-i\omega_b t} e^{+i\omega_s t} \\
&+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{-i\omega_b t} e^{-i\omega_d t} + \Gamma_{b-} H_{b-} \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{-i\omega_b t} e^{+i\omega_d t} \\
&+ \Gamma_{b-} H_{b-} h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{-i\omega_b t} e^{+2i\omega_b t} + \Gamma_{b-} H_{b-} h_{bz} \Gamma_{b+} H_{b+} e^{-i\omega_b t} e^{-2i\omega_b t}
\end{aligned}
$$

$$
\begin{aligned}
&= \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{a+} H_{a+} \right) e^{-i\left(2\omega_a + \omega_b\right)t} + \Gamma_{b-} H_{b-} \left(h_{az}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\left(2\omega_a + \omega_b\right)t} \\
&+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{bz} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{az}{}^{*} \Gamma_{a+} H_{a+} + h_{bz}{}^{*} \Gamma_{b+} H_{b+} \right) e^{-i\omega_b t} \\
&+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{b+} H_{b+} + h_{bz} \Gamma_{a+} H_{a+} \right) e^{-i\left(2\omega_b + \omega_a\right)t} + \Gamma_{b-} H_{b-} \left(h_{az}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a-}{}^{*} H_{a-}{}^{*} \right) e^{+i\omega_a t} \\
&+ \Gamma_{b-} H_{b-} \left(h_{az} \Gamma_{b-}{}^{*} H_{b-}{}^{*} + h_{bz}{}^{*} \Gamma_{a+} H_{a+} \right) e^{-i\omega_a t} + \Gamma_{b-} H_{b-} \left(h_{bz} \Gamma_{a-}{}^{*} H_{a-}{}^{*} + h_{az}{}^{*} \Gamma_{b+} H_{b+} \right) e^{-i\left(2\omega_b - \omega_a\right)t} \\
&+ \Gamma_{b-} H_{b-} h_{bz}{}^{*} \Gamma_{b-}{}^{*} H_{b-}{}^{*} e^{+i\omega_b t} + \Gamma_{b-} H_{b-} h_{bz} \Gamma_{b+} H_{b+} e^{-3i\omega_b t}
\end{aligned} \tag{91}
$$

7. Conclusion

This report was written with the intention of archiving the results of lengthy calculations needed in order to evaluate the feasibility of remote detection of magnetic systems by FMR. More specifically, these calculations make it possible to calculate the induced magnetization within a ferromagnetic body illuminated by electromagnetic waves at levels of excitation high enough to give rise to a nonlinear response in the body and induce it to radiate at new frequencies that are detected at a remote receiving antenna. The results tabulated here contribute to the study of nonlinear radar and target signatures in general.

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